SECTIONAL CURVATURE IN PIECEWISE LINEAR MANIFOLDS

BY DAVID A. STONE

Communicated by William Browder, February 13, 1973

A metric complex $M$ is a connected, locally-finite simplicial complex linearly embedded in some Euclidean space $R^l$. Metric complexes $M$ and $M'$ are isometric if they have subdivisions $L$ and $L'$ and if there is a simplicial isomorphism $h: L \to L'$ such that for every $a \in L$, the linear map determined by $h: a \to h(a)$ is an isometry (that is, it extends to an isometry of the affine spaces generated by these simplexes). This note is concerned with certain characteristics of a metric complex $M$ which are intrinsic, i.e., which depend only on the isometry class of $M$. The basic such characteristic is the intrinsic metric, which is best described in the piecewise linear context by H. Gluck [3]; for a more general treatment see W. Rinow [8].

Let $M \subseteq R^l$ be a metric complex and let $p$ be a point of $M$. Then the tangent cone $T_pM$ of $M$ at $p$ is defined to be the infinite cone with vertex $p$, generated by link($p$, $M$). The isometry class of $T_pM$ is intrinsic to $M$, for each $p$. An infinite ray $p\overrightarrow{x}$ in $T_pM$ will be called a tangent direction at $p$ to $M$.

Let $N_pM$ be a subcone of $T_pM$ and let $j$ be a nonnegative integer. Let $R^j \times N_pM$ be given the metric in which its factors are orthogonal. For various choices of $N_pM$ and $j$, $R^j \times N_pM$ will be isometric to $T_pM$. For example if $p$ is in the interior of a $j$-simplex of $M$, such a factoring exists. Consider those factoring of $T_pM$ for which $j$ is maximal; then the corresponding $N_pM$ are all isometric. Such an $N_pM$ will be called the normal geometry of $p$ in $M$, and denoted $v_pM$. For example, if $M$ is an $n$-manifold and $p$ is in the interior of an $(n - 1)$- or $n$-simplex, then $v_pM = \{p\}$. If $M$ is a 2-manifold, then $v_pM = \{p\}$ unless $p$ is a vertex of nonzero curvature, when $v_pM = T_pM$.

Clearly $j$ and $v_pM$ determine the metric geometry of $M$ near $p$.

For any $p \in M$ and any tangent direction $p\overrightarrow{x}$ at $p$ lying in $v_pM$ I have defined numbers $k_+(p\overrightarrow{x})$ and $k_-(p\overrightarrow{x})$, with $k_+(p\overrightarrow{x}) \leq k_-(p\overrightarrow{x})$, called the maximum and minimum curvatures of $M$ at $p$ in the direction $p\overrightarrow{x}$. The definitions are too long to give here. Roughly speaking, $k_-(p\overrightarrow{x})$ equals: $2\pi$ minus twice the maximum “angle” that can occur between $p\overrightarrow{x}$ and any other $p\overrightarrow{y} \leq v_pM$ as $y$ varies; $k_+(p\overrightarrow{x})$ is defined similarly, using a

AMS (MOS) subject classifications (1970). Primary 53C70, 57C99; Secondary 53C45.

1 Research supported in part by NSF Grants P22927 and P029431000.
mini-max. If $M$ is a 2-manifold, then $k_+(p\bar{x}) = k_-(p\bar{x})$ and depends only on $p$; they are both equal to the standard piecewise linear curvature of $M$ at $p$ (see Aleksandroff and Zalgaller [1] or W. Rinow [8]). There seems to be some connection between $k_-(p\bar{x})$ and, in the smooth case, the minimum sectional curvature at a point of two-planes containing a given tangent vector at that point; likewise between $k_+(p\bar{x})$ and the maximum such sectional curvature. To support this intuition I offer these results:

**Theorem 1.** Let $M$ be a complete metric complex such that $k_+(p\bar{x}) \leq 0$ for all $p \in M$ and all tangent directions $p\bar{x} \subseteq v_p M$. Then:

(i) for any $p, q \in M$ and any homotopy class $\psi$ of paths from $p$ to $q$ there is exactly one shortest path in $\psi$;

(ii) in particular, if $M$ is simply connected, then it is contractible;

(iii) if $M$ is a simply-connected manifold without boundary of dimension $n \geq 6$, then $M$ is piecewise linearly isomorphic to Euclidean space $\mathbb{R}^n$.

Theorem 1 is analogous to a theorem proved for smooth manifolds by E. Cartan [2] under the hypothesis that every sectional curvature be $\geq 0$.

**Theorem 2.** Let $M$ be a complete metric complex which is an $n$-manifold without boundary. Assume that whenever $a$ is an $(n - 2)$-simplex, whenever $p \in \text{int } a$ and whenever $p\bar{x} \subseteq v_p M$, then $k_-(p\bar{x}) \geq 0$. Then:

(i) if $n$ is even and $M$ orientable, then $M$ is simply connected;

(ii) if $n$ is odd, then $M$ is orientable.

In the smooth case a theorem analogous to (i) was proved by J. Synge [10], and (ii) is an elementary consequence of his method observed by A. Preissman [7].

**Theorem 3.** Let $M$ be a complete metric complex which is an $n$-manifold without boundary. Assume:

1. there is a number $k \geq 0$ such that whenever $a$ is an $(n - 2)$-simplex, whenever $p \in \text{int } a$ and whenever $p\bar{x} \subseteq v_p M$, then $\dim v_p M = 2$ and $k_-(p\bar{x}) \geq k$;

2. there is a number $Q$ such that whenever $a$ is an $n$-simplex of $M$ and $M$ is represented as a linear complex in $\mathbb{R}^l$, then the $n$-sphere in $\mathbb{R}^l$ that passes through the vertices of $a$ has radius $\leq Q$. Then:

(i) $M$ is compact (I can in fact give a crude estimate for the diameter of $M$);

(ii) $M$ has positive curvature "everywhere": $k_-(p\bar{x}) \geq 0$ provided that $p$ is not in the interior of an $(n - 1)$- or $n$-simplex.

Theorem 3 is a weak analogue of a theorem proved for smooth manifolds by S. Myers [6] under the hypothesis that the mean curvature be everywhere bounded above 0. I suspect that the curvature hypothesis of
Theorem 3 can be weakened once one has the right piecewise linear notion of mean curvature.

An amusing consequence of Theorem 3 is:

**THEOREM 4.** Let \( K \) be a simplicial 3-manifold without boundary. Assume that every 1-simplex is a face of at most five 3-simplexes. Then \( K \) is finite.

The proof is to give \( K \) a metric by making all the tetrahedra regular of side length 1; then the hypotheses of Theorem 3 are satisfied. A. Phillips has pointed out to me that \( \mathbb{R}^3 \) can be triangulated so that every 1-simplex is a face of at most six 3-simplexes.

**DISCUSSION OF THEOREM 1.** The proof of this theorem is analogous to the proof of Cartan’s theorem in the smooth case (see J. Milnor’s [5]). The curvature hypothesis on \( M \) is equivalent to the hypothesis that \( M \) has unique geodesics locally. This means: every \( p \in M \) has a neighbourhood \( U \) such that whenever \( x, y \in U \), then there is a unique geodesic in \( M \) from \( x \) to \( y \). Hence for any \( p, q \in M \) one can approximate (as in [5]) the space \( \Omega \) of paths from \( p \) to \( q \) and the energy function \( E: \Omega \to \mathbb{R}^1 \) by a finite-dimensional space \( V \) and a function \( F: V \to \mathbb{R}^1 \). \( F \) is not smooth; nonetheless one can show that \( F \) has no “critical points” except local minima. Conclusion (i) follows, as in [5].

In the smooth case one proves (iii) by inferring that at any point \( p \in M \) the exponential map \( \exp_p : T_p M \to M \) is globally defined and is a diffeomorphism. In the piecewise linear case this argument fails, even for 2-manifolds. However one can consider the function distance-from-\( p \rho_p : M \to \mathbb{R}^1 \) and verify that its only “critical point” is \( p \). It follows from a theorem of J. Stallings [9] (in the piecewise smooth context) that \( M \) is piecewise diffeomorphic to \( \mathbb{R}^n \), and hence from triangulation theory (see M. Hirsch and B. Mazur [4]) that \( M \) is piecewise linearly isomorphic to \( \mathbb{R}^n \). At one point in this argument the \( h \)-cobordism theorem is used to show that certain points are not “critical”; hence the restriction \( n \geq 6 \).

**DISCUSSION OF THEOREM 3.** (The proof of Theorem 2 is quite similar to that of Theorem 3.) The first (curvature) hypothesis on \( M \) implies that the whole \((n - 2)\)-skeleton \( M^{n-2} \) is intrinsic to \( M \), for it is the coarsest possible triangulation of the “singular set” of \( M \)—that is, of the set of points where the normal geometry is nontrivial. The second hypothesis then says that the singular set is “fairly dense” in \( M \); it implies for example that every point of \( M \) is distant at most \( Q \) from the singular set.

Let \( P \) be a number \( \geq Q \). Let \( a \) be a linear simplex in \( \mathbb{R}^l \) which satisfies hypothesis 2. Let \( S \) be an \((l - 1)\)-sphere with centre \( C \) and radius \( P \) which passes through the vertices of \( a \). Then \( C \) does not lie in the affine plane spanned by \( a \), so I can project \( a \) into \( S \) from \( C \). Call the image \( a# \); then \( a# \) is the \( P \)-spherical simplex associated to \( a \). Let \( M \) be the simplicial
complex $M$ re-metrized by replacing each $a \in M$ by the associated $P$-spherical simplex.

The proof of Theorem 3 now falls into four parts. First, whenever $P$ is large enough, then $\mathcal{M}$ satisfies hypothesis 1 (with a different bound $k \# \geq 0$ for the curvature). Second, one shows by induction on dim $\nu_p M$ that conclusion (ii) holds for $M$ and for $\mathcal{M}$. The inductive step is based on the third part, assumed proved in dimensions $\leq n$. The third part is to show that then $\mathcal{M}$ has diameter $\leq \pi P$. Finally, one has to compare the intrinsic metrics on $M$ and $\mathcal{M}$.

The nub of the proof is the third part. It is proved by inferring from hypothesis 1 for $\mathcal{M}$ that any geodesic $\alpha$ in $\mathcal{M}$ meets the singular set $\mathcal{M}^{n-2}$ at most in the endpoints of $\alpha$. Hence a neighbourhood of $\alpha$ can be immersed isometrically in the standard $n$-sphere $S$ of radius $P$. If $\alpha$ has length $\geq \pi P$, then its image $\alpha'$ in $S$, having the same length as $\alpha$, can be approximated by shorter paths $\beta'$ with the same endpoints. But any $\beta'$ close enough to $\alpha'$ corresponds to a path $\beta$ in $\mathcal{M}$ with the same endpoints as $\alpha$ and the same length as $\beta'$. Thus $\alpha$ is not a shortest path; this proves the assertion.

REFERENCES


Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11790