SELFCOMMUTATORS OF MULTICYCLIC HYPONORMAL OPERATORS ARE ALWAYS TRACE CLASS

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1. For $A$, $B$ operators on the Hilbert space $H$, $[A, B] = AB - BA$. The selfcommutator of $A$ is $[A^*, A]$. If $E$ is a closed proper subset of the plane, $R(E)$ will be the rational functions analytic on $E$. The operator $A$ is said to be $n$-multicyclic if there are $n$ vectors $g_1, \ldots, g_n \in H$, called generating vectors, such that $\{r(A)g_i : r \in R(sp(A)), 1 \leq i \leq n\}$ has span dense in $H$. This paper will outline a circle of ideas culminating in the following result.

**MAIN THEOREM.** If $A$ is an $n$-multicyclic hyponormal operator, then $[A^*, A]$ is in trace class, and $\text{tr}[A^*, A] \leq (n/\pi)\omega(sp(A))$, where $\omega$ is planar Lebesgue measure.

This result is especially interesting because of the scarcity of known conditions insuring that the selfcommutator lie in trace class. The above result is new even when $A$ is subnormal and has a cyclic vector in the usual sense. The best previous result in this direction is due to T. Kato [1], and states that if $\text{Re}(A)$ has finite spectral multiplicity $n$, then $[A^*, A]$ is in trace class. Kato provides a trace estimate which Putnam [4] is able to use to prove the above estimate, where $n$ is an upper bound for the spectral multiplicity of $\text{Re}(A)$.

The Kato-Putnam estimate and the main theorem above are independent. For example, using a result of J. W. Helton and R. Howe, unpublished as yet, which provides a lower bound for the spectral multiplicity of the real part of a hyponormal operator, one can see that the real part of the $1$-multicyclic operator given by multiplication by $z$ on $\mathbb{R}^2$ of a Swiss cheese has infinite spectral multiplicity almost everywhere.

Throughout the following, a space and the orthogonal projection onto that space will be denoted by the same symbol. All spaces are Hilbert spaces.

2. The following lemma is central.

**STRUCTURE LEMMA.** Let $T$ and $A$ be hyponormal operators on $H$ and $K$...
respectively, and let \( W: H \to K \) be a trace class operator with dense range, such that \( WT = AW \). Then \( \text{tr}[A^*, A] \leq \text{tr}[T^*, T] \).

**Proof.** It may be assumed that \( \text{tr}[T^*, T] < \infty \). Let \( N \) be the null space of \( W \). Since \( N \) is an invariant space for \( T \), \( TN \) is also hyponormal. It will be shown that \( \text{tr}[A^*, A] + \text{tr}[NT^*, TN] \leq \text{tr}[T^*, T] \).

Let \( \{\varphi_n\}_n \) be a complete orthonormal system of eigenvectors for \( W^*W \), with \( W^*W\varphi_n = \lambda_n^2\varphi_n, \lambda_n \geq 0 \). Then the vectors \( \{\psi_n: \lambda_n > 0\} \) given by \( W\varphi_n = \lambda_n\psi_n \) are a complete orthonormal basis for \( K \). Let \( L_t = H \oplus K \) have the norm \( \|h \oplus k\|^2 = t^2\|h\|^2 + \|k\|^2 \), for \( t > 0 \), and let \( J \) be the closed subspace spanned by the vectors \( \{h \oplus Wh: h \in H\} \).

\[
\{(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\}_n
\]
is a complete orthonormal basis for \( J \). Note that \( J \) is an invariant space for \( T \oplus A \), so \( (T \oplus A)J \) is hyponormal. \((T \oplus A)H = (T \oplus 0)\), which, when restricted to \( H \oplus 0 \), is unitarily equivalent to \( T \), so it can be shown that \( H - J \) is in trace class, \([J(T \oplus A)^*, (T \oplus A)J]\) will lie in trace class, and

\[
\text{tr}[J(T \oplus A)^*, (T \oplus A)J] = \text{tr}[H(T \oplus A)^*, (T \oplus A)H] = \text{tr}[T^*, T].
\]

But the space spanned by the vectors \( \{\varphi_n, \psi_n\} \) reduces \( H - J \), and on this space \( H - J \) has trace norm \( 2\lambda_n(t^2 + \lambda_n^2)^{-1/2} \). Thus, \( H - J \) has trace norm \( \sum_n 2\lambda_n(t^2 + \lambda_n^2)^{-1/2} \leq 2t^{-1} \sum_n \lambda_n \). Now consider

\[
\text{tr}[J(T \oplus A)^*, (T \oplus A)J]
\]

\[
= \sum_{\lambda_n > 0} \{\|(T \oplus A)(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\|^2
\]

\[
- \|J(T^* \oplus A^*)(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\|^2\}
\]

\[
+ \sum_{\lambda_n = 0} \{\|(T \oplus A)(t^{-1}\varphi_n \oplus 0)\|^2 - J(T^* \oplus A^*)(t^{-1}\varphi_n \oplus 0)\|^2\}.
\]

The diligent reader will discover that the summand in the first sum approaches \( \|A\varphi_n\|^2 - \|A^*\varphi_n\|^2 \) as \( t \to 0 \). (To show that \( \|J(0 \oplus u)\|^2 \to \|u\|^2 \), he will evaluate the norm of the projection using the orthonormal basis for \( J \), and apply the Lebesgue monotone convergence theorem to the resulting sum.) A similar technique, applied to the summands of the second sum, and now invoking Lebesgue dominated convergence, shows that they approach

\[
\|T\varphi_n\|^2 - \sum_{\lambda_m = 0} \{\langle T^*\varphi_n, \varphi_m \rangle^2\} = \{TN\varphi_n\|^2 - \|NT^*\varphi_n\|^2\}.
\]

Thus, by Fatou’s theorem, \( \text{tr}[A^*, A] + \text{tr}[NT^*, TN] \leq \text{tr}[T^*, T] \).
In light of the Structure Lemma, it is obviously desirable to produce a supple family of hyponormal operators $T$ with trace class selfcommutators.

**Definition.** For $\mu$ a finite measure with compact support $E$ contained in the compact set $F$, $R^2(F, \mu)$ will be the closure of $R(F)$ in $L^2(\mu)$. $R^2(E, \mu)$ will be written $R^2(\mu)$. If $F$ does not divide the plane, $R^2(F, \mu) = H^2(\mu)$. $T_f$ on $R^2(F, \mu)$ will be the operator $PL_fP$, where $P$ is the orthogonal projection on $L^2(\mu)$ with range $R^2(F, \mu)$.

**Computational Lemma.** Let $D = \{z: |z| < 1\}$, and let $H = H^2(\chi_D \omega)$. For $f \in H^2(\chi_D \omega)$, let $T_f = L_f$ on $H$, where $L_f$ is the Laurent operator. If $f = \sum_{n=0}^{\infty} a_n z^n$, then

$$\text{tr}[T^*_f, T_f] = \sum_{n=1}^{\infty} n |a_n|^2 = \frac{1}{\pi} \int |f'|^2 \, d\omega$$

$$= \pi^{-1} \{\text{Area of } f(D), \text{counting the multiplicity of the covering}\}.$$

**Proof.** The first equality may be computed directly, using the basis $\{(n + 1)^{1/2} z^n\}_{n=0}^{\infty}$. The others are well known.

**Corollary.** Let $U$ be a simply connected open set with a smooth Jordan curve for its boundary. Let $g$ be the Riemann map from $U$ to $D$. Then the map $T_z$ on $H^2(\chi_U \omega)$ satisfies $\text{tr}[T^*_z, T_z] = \pi^{-1} \omega(U)$.

**Proof.** Taking $g^{-1} = f$, $T_z$ is unitarily equivalent to $T_f$ above.

**Remark.** If $A_1, \ldots, A_n$ are each $T_z$ on the respective spaces $R^2(\mu_i)$, if their spectra are pairwise disjoint and if $\text{tr}[A^*_i, A_i] = \rho_i < \infty$, then the operator $T_{z_1} + \cdots + T_{z_n}$ satisfies $\text{tr}[T^*_z, T_z] = \rho_1 + \cdots + \rho_n$.

It is also necessary to produce trace class intertwining maps. Let $T \in B(H)$. Suppose there is a map $z \rightarrow k_z$, from the open set $U$ to $H$, which is conjugate analytic as a map into $H$ in the strong topology, and such that there is a vector $x \in H$ satisfying $\langle r(T)x, k_z \rangle = r(z)$, for all rational functions $r$ with poles off $\text{sp}(T)$, and all $z \in U$. Then the triple $(U, k_z, x)$ will be called an analytic evaluation for $T$, if $T^* k_z = \bar{z} k_z$ for all $z \in U$.

**Intertwining Lemma.** Let $(U, k_z, x)$ be an analytic evaluation for $T \in B(H)$, and suppose that $x$ is a $1$-multicyclic vector for $T$. If $u \in H$, let $\hat{u}(z) = \langle u, k_z \rangle$, for $z \in U$. Let $A \in B(K)$ such that $\text{sp}(A) \subseteq U$, and let $y \in K$. Define $W: H \rightarrow K, W u = \hat{u}(A)y$. Then $WT = AW$, and $W$ lies in trace class.

**Proof.** $\hat{u}$ is analytic on an open neighborhood of $\text{sp}(A)$, and so $\hat{u}(A)$ is well defined, say by the Riesz integral. Since $k_z$ is an eigenvector for $T^*$ with eigenvalue $\bar{z}$, $(Tu)^* = \bar{z} u$. Thus $WT = AW$. That $W$ lies in trace class results from the fact that the map $z \rightarrow k_z$ is strongly conjugate analytic on
an open neighborhood of \( \text{sp}(A) \). Let \( \Gamma_1 \) be a finite set of smooth Jordan curves bounding \( \text{sp}(A) \) from \( U^c \), and let \( \Gamma_2 \) be another such set bounding \( \Gamma_1 \) from \( U^c \), and \( \Gamma_3 \) a third, bounding \( \Gamma_2 \) from \( U^c \). Let \( \lambda \) be arc length on \( \Gamma_i \). Let \( H_i \) be the closure of the functions \( \{ \hat{u} : u \in H \} \) in \( L^2(\lambda_i) \). Let \( W_3: H \to H_3 \) by \( W_3u = \hat{u}|_{\Gamma_3} \), \( H_3 \), \( H_2 \), and \( H_1 \) admit analytic evaluations. Define \( W_iu = \hat{u}|_{\Gamma_i} \) for \( u \in H_i \) for \( i = 2, 1 \) and \( W_0u = \hat{u}(A)y \) for \( u \in H_1 \). \( W = W_0W_1W_2W_3 \), each \( W_i \) is bounded and it is easy to represent \( W_2 \) and \( W_1 \) as integral operators with square-summable kernels. Thus \( W_2 \) and \( W_1 \) are Hilbert-Schmidt operators, and so \( W_2W_1 \) is in trace class \( [2] \).

**Corollary.** Let \( \mu \) be a finite measure with compact support. Let \( K = H^2(\mu) \) and let \( E \) be the complement of the unbounded component of the complement of \( \text{sp}(T_2) \). \([T^*_z, T_z] \) is in trace class and \( \text{tr}[T^*_z, T_z] \) is small.

**Proof.** Let \( A = T_z \) on \( K \). Let \( U \) be a simply connected open set with smooth Jordan boundary such that \( E \subset U \) and \( \omega(U) - \omega(E) \) is small. Let \( T \) be \( T_z \) on \( H = H^2(x|v2\gamma^z) \), where \( g \) is as in the corollary to the Computational Lemma. Then \( \text{tr}[T^*, T] = \pi^{-1} \omega(U) \). Since \( |g|^2 \) is bounded away from zero on compact sets in \( U \), there exist vectors \( k \) such that \( (U, k, 1) \) is an analytic evaluation for \( T \). Thus the Intertwining Lemma applies. \( W_1 = 1 \) is a cyclic vector for \( T_z \) on \( K \), so \( W \) has dense range. Thus, the Structure Lemma applies, and so \( \text{tr}[A^*, A] \leq \pi^{-1} \omega(U) \). Thus \( \text{tr}[A^*, A] \leq \pi^{-1} \omega(E) \).

**Subspace Dominance Lemma.** Let the hyponormal operator \( A \in B(H) \) be \( n \)-multicyclic, with generating vectors \( g_1, \ldots, g_n \). Let \( E \) be a compact set containing \( \text{sp}(A) \). Let \( V \) be the closure of the space spanned by \( \{ r(A)g_i : r \in R(E), 1 \leq i \leq n \} \). Then \( V \) is an invariant space for \( A \), \( AV \) is hyponormal, \( \text{sp}(AV) \subset E \), \( AV \) is \( n \)-multicyclic with generating vectors \( g_1, \ldots, g_n \) and \( \text{tr}[A^*, A] \leq \text{tr}[AV^*, AV] \).

**Proof.** Unless \( \text{tr}[VA^*, AV] < \infty \), there is nothing to prove. Let \( \{ a_i \}_{i=1}^\infty \) be a sequence of points in \( E \sim \text{sp}(A) \) which land densely in each component of \( \text{sp}(A) \) which lies entirely in \( E \). Let \( r_m(z) = \prod_{i=1}^m (z - a_i)^{-1} \). Let \( V = V_m, V_0 = V \). Then \( V_{m+1} \supseteq V_m \), rank \( (V_{m+1} - V_m) \leq n \), and \( V_m \supset H \) strongly. Thus \( \text{tr}[V_mA^*, AV_m] = \text{tr}[VA^*, AV] \). Let \( \{ e_k \}_{k} \) be an orthonormal basis for \( H \).

\[
\text{tr}[V_mA^*, AV_m] = \sum_k \left[ \| AV_m e_k \|^2 - \| V_m e_k \|^2 \right].
\]

Thus, since the summands are all nonnegative and approach the corresponding terms for \( \text{tr}[A^*, A] \), Fatou's lemma guarantees the desired inequality.

**Second Computational Lemma.** Let \( U_1, \ldots, U_n \) be open sets with
disjoint closures, each bounded by finitely many disjoint smooth Jordan curves. Let \( U = \bigcup_i U_i \) and \( H = R^2(\chi_U - \omega) \). Then \( T_z \) on \( H \) satisfies\( \text{tr}[T_z^*, T_z] \leq \pi^{-1} \omega(U) \).

**Proof.** Let \( \{G_i\}_{i=1}^m \) be simply connected open sets with smooth Jordan curves as boundaries such that each \( G_i \) lies in a separate bounded component of \( U^{-\varepsilon} \), and such that \( \sum_i \omega(G_i) \) is close to the total area of the bounded components of \( U^{-\varepsilon} \). Choose \( g_i \) so that \( T_z \) on \( H^2(|g_i|^2 \chi_{G_i}, \omega) \) satisfies \( \text{tr}[T_z^*, T_z] = \pi^{-1} \omega(G_i) \). Let \( T \) be \( T_z \) on \( H, S \) be \( T_z \) on \( R^2(XUCD) \).

Let \( T \) be \( T_z \) on \( H^2(\chi_U - \omega + \sum_i |g_i|^2 \chi_{G_i}, \omega) \), \( T \) be \( T_z \) on \( H^2(\chi_U - \omega + \sum_i |g_i|^2 \chi_{G_i}, \omega) \).

Let \( U \) be an open set bounded by a finite number of disjoint smooth Jordan curves, such that \( \text{sp}(A) \subset U \), and \( \omega(U) - \omega(\text{sp}(A)) \) is small. Let \( K' \) be the space spanned by \( \{r(A)g_i : r \in R(U^-) \} \) and \( 1 \leq i \leq n \). Let \( A' \) be the restriction of \( A \) to \( K' \). \( A' \) is hyponormal, and \( \text{sp}(A') \subset U \).

Let \( W : R^2(\chi_U - \omega) \to K' \) be \( K' \)-valued analytic, and the triple \( (U, k_z, 1) \) is an analytic evaluation. Thus by the Intertwining Lemma, \( \text{tr}[A^*, A] \leq (n/\pi) \omega(\text{sp}(A)) \).

**Theorem 1.** Let \( A \in B(K) \) be hyponormal, with \( n \)-multicyclic generating vectors \( g_1, \ldots, g_n \). Then \( \text{tr}[A^*, A] \leq (n/\pi) \omega(\text{sp}(A)) \).

**Proof.** Let \( U \) be an open set bounded by a finite number of disjoint smooth Jordan curves, such that \( \text{sp}(A) \subset U \), and \( \omega(U) - \omega(\text{sp}(A)) \) is small. Let \( K' \) be the space spanned by \( \{r(A)g_i : r \in R(U^-) \} \) and \( 1 \leq i \leq n \). Let \( A' \) be the restriction of \( A \) to \( K' \). \( A' \) is hyponormal, and \( \text{sp}(A') \subset U \).

Let \( A' \) be a set of \( n \)-multicyclic vectors for \( A' \). By the Subspace Dominance Lemma, \( \text{tr}[A^*, A] \leq \text{tr}[A'^*, A'] \).

Let \( T = \bigoplus \sum_{i=1}^n T_z \) acting on \( H = \bigoplus \sum_{i=1}^n R^2(\chi_U) \).

Let \( T_z \) be \( T_z \) on \( H^2(\chi_U - \omega) \). By the Second Computational Lemma, \( \text{tr}[T^*, T] \leq (n/\pi) \omega(U) \). Thus, it only remains to produce an intertwining map between \( T \) and \( A' \) satisfying the conditions of the Structure Lemma.

\( R^2(\chi_U - \omega) \) has reproducing kernel \( k_z \) at each \( z \in U \). The map \( z \to k_z \) is strongly conjugate analytic, and the triple \( (U, k_z, 1) \) is an analytic evaluation. Thus by the Intertwining Lemma, the map \( W_i : R^2(\chi_U - \omega) \to K' \) defined by \( Wf = \tilde{f}(A')g_i \) lies in trace class, and \( W_i T_z = A' W_i \).

Let \( W : \bigoplus \sum_{i=1}^n R^2(\chi_U - \omega) \to K' \) by \( W = \sum_{i=1}^n W_i \). \( W \) lies in trace class, and \( WT = A' W \). Clearly, the range of \( W \) is dense in \( K' \). Thus\( \text{tr}[A^*, A] \leq \text{tr}[A'^*, A'] \leq \text{tr}[T^*, T] \leq (n/\pi) \omega(U) \).
COROLLARY (PUTNAM'S THEOREM [3]). If $A \in B(H)$ is hyponormal, then $\|A^*, A\| \leq \pi^{-1} \omega(\text{sp}(A))$.

**Proof.** Let $x \in H$, $\|x\| = 1$, and let $V$ be the closure of the set of vectors $\{r(A)x : r \in \mathcal{R}(\text{sp}(A))\}$. $V$ is an invariant space for $A$. Let $A'$ be the restriction of $A$ to $V$. $A'$ is hyponormal.

If $y \in V$ and $a \in \text{sp}(A)'$, $(A - al)^{-1}y \in V$. Thus $\text{sp}(A) \supset \text{sp}(A')$. It is clear that $A'$ is 1-multicyclic. Thus

$$\langle [A^*, A]x, x \rangle = \|Ax\|^2 - \|A^*x\|^2 \leq \|Ax\|^2 - \|VA^*x\|^2$$

$$= \|Ax'\|^2 - \|A'^*x\|^2$$

$$= \langle [A'^*, A']x, x \rangle \leq \text{tr}[A'^*, A']$$

$$\leq \pi^{-1} \omega(\text{sp}(A')) \leq \pi^{-1} \omega(\text{sp}(A)).$$

3. The techniques used above suffice to yield the following results.

**Theorem 2.** If the hyponormal operator $A$ has analytic evaluation $(U, k_z, x)$, then $\text{tr}[A^*, A] \geq \pi^{-1} \omega(U)$.

**Theorem 3.** If $A$ is a 1-multicyclic hyponormal operator with generating vector $x$, if $V$ is an invariant space for $A$ containing $x$, and if $A'$ is the restriction of $A$ to $V$, then

$$\text{tr}[A^*, A] + \pi^{-1} \omega(\text{sp}(A') \sim \text{sp}(A)) \leq \text{tr}[A'^*, A'].$$

The corresponding result for $n$-multicyclic hyponormal operators is rather more complicated, and requires a fairly lengthy explanation.

**Theorem 4.** For $r \in \mathcal{R}(E)$, $T_r$ on $R^2(E, \mu)$ satisfies

$$\left[T_r^*, T_r\right] \leq \frac{1}{\pi} \int_{\text{sp}(T_r)} |r|^2 \, d\omega.$$

Note that the quantity $[T_r^*, T_r]$ is a quadratic norm on $\mathcal{R}(E)$. The above theorem may be generalized to all functions in the Hilbert space so determined. The following is unknown.

**Conjecture.** There is a measurable function $g$ defined on $\text{sp}(T_r)$ such that $0 \leq g \leq 1$, and $\text{tr}[T_r^*, T_r] = \pi^{-1} \int_{\text{sp}(T_r)} |r|^2 \, g \, d\omega$ for all $r \in \mathcal{R}(E)$.

**Theorem 5.** If $R^2(E, \mu)$ has analytic evaluation $(U, k_z, 1)$, $F$ is a compact subset of $U$, $v$ is a finite measure supported on $F$, and $r \in \mathcal{R}(E)$, then $\text{tr}[T_r^*, T_r]$ is the same, whether computed on $R^2(E, \mu)$ or on $R^2(E, \chi_{F-\mu} + v)$.

**Theorem 6.** If $R^2(E, \mu)$ has analytic evaluation $(U, k_z, 1)$, and $0 \leq g \leq 1$ is a measurable function such that $g^{-1}([0, 1]) \subset U$, then for all $r \in \mathcal{R}(E)$, $\text{tr}[T_r^*, T_r]$ is not increased when it is computed on $R^2(E, g\mu)$ rather than on $R^2(E, \mu)$. 
THEOREM 7. Let $A^2(U)$ be the Hilbert space of all functions analytic on the open set $U$, and square summable with respect to $\chi_{U\omega}$. Let $f$ be bounded and analytic on $U$. Then $\text{tr}[T^*, T_f] = \pi^{-1} \int_{f(U)} \eta(z, f) \, d\omega$, where $\eta(z, f)$ is the cardinality of $f^{-1}(z)$.

This theorem may be generalized to the setting of complex manifolds. For a finite measure with compact support $E$, and $F$ a compact set containing $E$, let $R = R^2(F, \mu) \subseteq L^2(\mu)$, and for $f \in L^\infty(\mu)$, define the "Hankel operator" $H_f$ by $H_f = (I - R)L_fR$. Let $\mathcal{H} = \{f \in L^\infty(\mu) : H_f \text{ is compact}\}$.

THEOREM 8. If $f \in R(F)$, then $H_f$ is a Hilbert-Schmidt operator. $\mathcal{H}$ is a closed subalgebra of $L^\infty(\mu)$, and $\mathcal{H}$ contains $L^\infty(\mu) \cap R^2(F, \mu) + C(E)$.

REFERENCES