1. The aim of this paper is to study the nonlinear differential equation

\[ Ex = Nx \]

where \( N \) is a nonlinear operator in a real Hilbert space \( S \), and \( E \) is a linear differential operator in \( S \) with preassigned linear homogeneous boundary conditions. The idea is to reduce the problem to a finite dimensional setting and this technique has been used by several authors. We use here a method due to Cesari [4]. This method has been extensively developed in the existence analysis of differential equations by Cesari, Hale, Locker, Mawhin and others. For a detailed bibliography one is referred to Cesari [5].

In this paper, by applying results from the theory of monotone operators, we show that, under suitable monotonicity hypotheses on \( N \), the equation \( Ex = Nx \) can be solved. In the present short presentation we restrict ourselves to the simplest hypotheses on \( E, N \) and \( S \), even though the results obtained here hold under more general conditions.

2. Let \( S \) be the direct sum of the subspaces \( S_0 \) and \( S_1 \) and let \( P:S \to S_0 \) be a projection operator with null space \( S_1 \) and \( H:S_1 \to S_1 \) a linear operator such that (h1) \( H(I - P)Ex = (I - P)x \), \( x \) belonging to the domain of \( E \). If \( y \) is a solution of (1), then \( Ey = Ny \) implies \( H(I - P)Ey = H(I - P)Ny \). Hence, \( (I - P)y = H(I - P)Ny \); and finally

\[ y = Py + H(I - P)Ny. \]

Thus, any solution of (1) is a solution of (2). If we also have that (h2) \( EPx = PEx \) and (h3) \( EH(I - P)Nx = (I - P)Nx \), then from (2) we derive

\[ Ey = EPy + EH(I - P)Ny = PEy + (I - P)Ny. \]

Hence, \( Ey - Ny = P(Ey - Ny) \). Thus, any solution \( y \) of (2) is a solution of (1) if and only if \( y \) satisfies

\[ P(Ey - Ny) = 0. \]
Thus, under hypotheses (h₁), (h₂), and (h₃), we have

**Theorem.** An element \( y \) in \( S \) is a solution of (1) if and only if \( y \) is a solution of equations (2) and (3).

Equations (2) and (3) are called the auxiliary and bifurcation equations respectively. Note that, if \( S \) is a separable Hilbert space with norm \( \| x \|^2 = \langle x \cdot x \rangle \) and \( \{ \phi_1, \phi_2, \ldots \} \) as an orthonormal basis, and we consider \( S_0 \) as spanned by \( \{ \phi_1, \phi_2, \ldots, \phi_m \} \), then (3) reduces to the finite system of equations \( (Ey - Ny) \cdot \phi_i = 0, \ i = 1, 2, \ldots, m \).

3. Let us assume that the associated linear problem \( Ex + \lambda x = 0 \) (with preassigned linear homogeneous boundary conditions) has a countable system of eigenvalues \( \lambda_i \) and eigenfunctions \( \phi_i \) such that \( \lambda_i \leq \lambda_{i+1}, \lambda_i \to +\infty \) as \( i \to \infty \) and \( \{ \phi_i \} \) is a complete orthonormal system in the Hilbert space \( S = L_2(A) \) of all square integrable functions \( x(a), a \in A \).

Any element \( x \in S \) can be written as \( \sum c_i \phi_i \).

Let \( Px = \sum c_i \phi_i \) and \( Ex = -\sum c_i \lambda_i \phi_i \). Thus,

\[
(I - P)x = \sum_{m+1}^{\infty} c_i \phi_i, \quad (I - P)x \in S_1,
\]

and for \( m \) such that \( \lambda_{m+1} > 0 \), let \( H : S_1 \to S_1 \) be defined by \( H(I - P)x = -\sum_{m+1}^{\infty} c_i \lambda_i^{-1} \phi_i \).

It can be easily seen that \( H(I - P)Ex = (I - P)x, \ EPx = PEx, \ EH(I - P)x = (I - P)x \).

For \( x = \sum c_i \phi_i \), we have

\[
\langle -H(I - P)x, x \rangle = \sum_{m+1}^{\infty} c_i^2 \lambda_i^{-1} \geq \lambda_{m+1} \sum_{m+1}^{\infty} c_i^2 \lambda_i^{-2} = \lambda_{m+1} \| H(I - P)x \|^2.
\]

Hence, the operator \( -H(I - P) \) is a linear, monotone operator. Since it is bounded, it is maximal monotone.

We now use the Theorem above to solve (1). To this end, we have to solve (2) and (3) respectively. Let us first consider the auxiliary equation (2), i.e., \( y = Py + H(I - P)Ny \). Let \( x^* \) be any element of \( S_0 \) and consider the equation

\[
y - H(I - P)Ny = x^*.
\]

This equation is of the type \( u + LNu = x^* \), where \( L \) is a (linear) maximal monotone operator; it has been studied by Browder [2], Brezis [1], Kolodner [7] and several others, where \( N \) is assumed to satisfy suitable monotonicity hypotheses.
In view of the fact that $\langle -H(I - P)x, x \rangle \geq \lambda_{m+1} \| -H(I - P)x \|^2$ and applying the result of Hess [6], we conclude that (4) has always a unique solution $y^*$ for each $x^* \in S_0$, provided $N$ is also hemicontinuous.

We now proceed to consider the bifurcation equation (3). Thus we have to solve the equation $PNy^* = PEy^*$, where $y^*$ is the solution of (4) corresponding to $x^* \in S_0$. But $PEy^* = EPy^* = Ex^*$ and thus equation (3) reduces to

$$PN[I - H(I - P)N]^{-1}x^* - Ex^* = 0.$$ 

Let $M = N[I - H(I - P)N]^{-1}$. And let $u = Ma$, $v = Mb$, where $a, b \in S_0$. Then, $u = Np$, $v = Nq$, where $p = (I - H(I - P)N)^{-1}a$ and $q = (I - H(I - P)N)^{-1}b$. Thus

$$\langle u - v, a - b \rangle = \langle Np - Nq, a - b \rangle$$

$$= \langle Np - Nq, p - H(I - P)Np - q + H(I - P)Nq \rangle$$

$$= \langle Np - Nq, p - q \rangle$$

$$+ \langle Np - Nq, -H(I - P)Np + H(I - P)Nq \rangle.$$ 

The first term on the right hand is $\geq 0$ because $N$ is monotone and the second is so because $-H(I - P)$ is monotone. Hence,

$$PM = PN[I - H(I - P)N]^{-1}$$

treated as an operator from $S_0$ to $S_0$ is monotone, for if $a, b \in S_0$, then $\langle PMa - PMb, a - b \rangle = \langle Ma - Mb, a - b \rangle$. Further, if $a \in S_0$, then the equation

$$a = (I + PN[I - H(I - P)N]^{-1})x$$

reduces to $a = x + PNp$, where $p = [I - H(I - P)N]^{-1}x$, or


By arguing as before it can be shown that this equation is solvable for $p$, and thus it follows from (6) that we can find $x \in S_0$ such that (5) is solvable. Hence, $PN[I - H(I - P)N]^{-1}$ is maximal monotone over $S_0$, a finite dimensional space.

Thus we are reduced to an equation in the finite dimensional space $S_0$ of the form $Mx^* - Ex^* = 0$ where $M$ is maximal monotone. If $\langle Ex^*, x^* \rangle \leq 0$, as is the case when all the $\lambda_i$'s are $\geq 0$, then the above equation is solvable. If, however, $E$ has a finite number of negative eigenvalues, then one can proceed in several ways. Thus if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq 0$, then one can apply Browder [3, p. 21] and conclude: If $[I - H(I - P)N]^{-1}$ is continuous and

$$\langle Nx_1 - Nx_2, x_1 - x_2 \rangle \geq c\|x_1 - x_2\|^2, \quad c > -\lambda_1,$$

then the bifurcation equation is solvable.
REFERENCES


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