Introduction. Following Solovay [2], let ‘ZF’ denote the axiomatic set theory of Zermelo-Fraenkel and let ‘ZF + DC’ denote the system obtained by adjoining a weakened form of the axiom of choice, DC, (see p. 52 of [2] for a formal statement of DC). From DC a ‘countable’ form of the axiom of choice is obtainable. More precisely, if \( \{B_n : n \in \mathbb{N}\} \) is a countable collection of nonempty sets then it follows from DC that there exists a function \( f \) with domain \( \mathbb{N} \) such that \( f(n) \in B_n \) for each \( n \).

The system ZF + DC is important because all the positive results of elementary measure theory and most of the basic results of elementary functional analysis, except for the Hahn-Banach theorem and other such consequences of the axiom of choice, are provable in ZF + DC. In particular, the Baire category theorem for complete metric spaces and the closed graph theorem for operators between Fréchet spaces are provable in ZF + DC.

Solovay shows [2] that the proposition, *Each subset of the real numbers is Lebesgue measurable*, cannot be disproved in ZF + DC. He does this by constructing a model for ZF + DC in which the proposition becomes a true statement.

We shall see that the proposition, *Each linear operator on a Hilbert space is a bounded linear operator*, is consistent with the axioms of ZF + DC. Other results of this type are obtained. For example, *Whenever \( X \) and \( Y \) are separable Fréchet groups and \( h: X \to Y \) is a homomorphism then \( h \) is continuous*, cannot be proved or disproved in ZF + DC.

Fortunately all the hard work in model theory has been done by Solovay. All that we use here is straightforward functional analysis.

All operators on a Hilbert space are bounded. We recall that a subset \( S \) of a topological space \( T \) is said to have the Baire property if there exists an open set \( U \) such that \( (U \setminus S) \cup (S \setminus U) \) is meagre. Let BP be the proposition: *Each subset of a complete separable metric space has the Baire property*. In [2, §4], Solovay outlines an argument which shows that when BP is interpreted in his model for ZF + DC then it becomes a true statement. Hence BP is consistent with the axioms of ZF + DC provided Solovay’s model exists. We adjoin BP as an axiom and denote the extended system by ‘ZF + DC + BP’.

In this paper certain propositions will be shown to be theorems of ZF + DC + BP. It is easy to show, by a Hamel base argument, that for each such proposition its negation is a theorem in ZFC (ZF with the axiom of choice adjoined). So these propositions can neither be proved nor disproved in ZF + DC, provided Solovay's model exists.

Let I be the axiom: There exists an inaccessible cardinal. Solovay uses the hypothesis that there exists a (transitive) model for ZFC + I when constructing his model.

From now onward we work in ZF + DC + BP. All our theorems are derived in this system.

**Lemma 1.** Let X and Y be separable metric spaces and let X be complete. Let $f : X \to Y$ be any function mapping X into Y. Then there exists a meagre set $N \subseteq X$ such that the restriction of $f$ to $X \setminus N$ is continuous.

Choose $\varepsilon > 0$. Let $\{y_r : r = 1, 2, \ldots\}$ be a countable dense subset of Y. For each $r$, let $S_r$ be the open sphere centred on $y_r$ with radius $\varepsilon/2$. Then $Y = \bigcup_1^\infty S_r$.

Let $A_1 = S_1$ and, for $n \geq 1$, let $A_{n+1} = (\bigcup_1^{n+1} S_r) - (\bigcup_1^n S_r)$. So $Y = \bigcup_1^\infty A_n$, where each $A_n$ is contained in an open sphere of radius $\varepsilon/2$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let $B_n = f^{-1}[A_n]$ for $n = 1, 2, \ldots$. Then $X = \bigcup_1^\infty B_n$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.

For any $n$, $B_n$ has the Baire property and so there is an open set $U_n$ and a meagre set $M_n$, where $U_n = (B_n \setminus U_n) \cup (U_n \setminus B_n)$, such that $U_n \cap (X \setminus M_n) = B_n \cap (X \setminus M_n)$. Let $M$ be the meagre set $\bigcup_1^\infty M_n$. Then $U_n \cap (X \setminus M) = B_n \cap (X \setminus M)$ for each $n$. Thus $B_n \cap (X \setminus M)$ is an open subset of $X \setminus M$ in the relative topology of $X \setminus M$.

Let $J$ be the set of all natural numbers $n$ for which $B_n \cap (X \setminus M)$ is not empty. By DC there exists a function $\xi$ with domain $J$ such that $\xi(n) \in B_n \cap (X \setminus M)$ for each $n$. Let $h$ be the function defined on $X \setminus M$ by $h(x) = f(\xi(n))$ whenever $x \in B_n \cap (X \setminus M)$.

Let $(z_j) (j = 1, 2, \ldots)$ be any sequence in $X \setminus M$ which converges to a point $z$ in $X \setminus M$. Then, for some $n \in J$, $B_n \cap (X \setminus M)$ is an open neighbourhood of $z$ in the relative topology of $X \setminus M$. So there exists a natural number $k$ such that $z_j \in B_n \cap (X \setminus M)$ whenever $j \geq k$. Thus $h(z_j) = h(z)$ whenever $j \geq k$. So $h : (X \setminus M) \to Y$ is continuous. Whenever $x \in X \setminus M$ then $x \in B_n \cap (X \setminus M)$ for some $n \in J$ and thus

$$d(h(x), f(x)) = d(f(\xi(n)), f(x)) < \varepsilon.$$ 

By putting $\varepsilon = 1/m (m = 1, 2, \ldots)$ we can find a sequence of functions $(h_m) (m = 1, 2, \ldots)$ and a sequence of meagre sets $(N_m) (m = 1, 2, \ldots)$ such that $h_m$ is a continuous map of $X \setminus N_m$ into $Y$ and $d(h_m(x), f(x)) < 1/m$. 


for each $x \in X \setminus N$. Let $N$ be the meagre set $\bigcup_{m=1}^{\infty} N_m$. Then $(h_m)(m = 1, 2, \ldots)$ converges uniformly to $f$ on $X \setminus N$. So $f$ is continuous on $X \setminus N$.

**Theorem 2.** Let $X$ and $Y$ be separable metrizable topological groups and let $X$ be complete. Let $H : X \to Y$ be any group homomorphism. Then $H$ is continuous.

Let $(x_n) (n = 1, 2, \ldots)$ be a sequence in $X$ converging to a point $x$. By Lemma 1, there is a meagre set $M$ such that $H$ is continuous when restricted to $X \setminus M$.

By the Baire category theorem, which is valid in ZF + DC, there exists $z \in X$ such that $z$ is not in the meagre set $x^{-1}M \cup \bigcup_{n=1}^{\infty} (x_n^{-1}M)$. Thus $xz \in X \setminus M$ and $x_nz \in X \setminus M$ for each $n$. Hence $H(xz) = \lim H(x_nz)$. Since $H$ is a homomorphism, $H(z) = \lim H(x_n)$.

The elegant argument used in Theorem 2 is due to Banach, see Theorem 4, Chapter 1 [1]. I wish to thank Professor A. Wilansky for drawing my attention to this reference.

In the following we do not require Fréchet spaces to be locally convex.

**Theorem 3.** Let $X$ be any Fréchet space and let $Y$ be a separable metrizable topological vector space. Let $T : X \to Y$ be a linear map. Then $T$ is continuous.

Let $(x_n) (n = 1, 2, \ldots)$ be any sequence in $X$ which converges to zero. Let $X_0$ be the closed linear span of $\{x_n : n = 1, 2, \ldots\}$ so that $X_0$ is a separable Fréchet space. Then, by the preceding theorem, the restriction of $T$ to $X_0$ is continuous. Thus $Tx_n \to 0$ as $n \to \infty$. So $T$ is continuous.

**Corollary 4.** Each linear functional on a Fréchet space is continuous.

**Theorem 5.** Let $X$ and $Y$ be Fréchet spaces and let $T : X \to Y$ be a linear map. If there exist enough functionals on $Y$ to separate the points of $Y$ then $T$ is continuous.

Let $(x_n) (n = 1, 2, \ldots)$ be a sequence in $X$ converging to $x$ and suppose $(Tx_n) (n = 1, 2, \ldots)$ converges to $y$. For any functional $\phi$ on $Y$, $\phi$ is continuous on $Y$ and $\phi T$ is continuous on $X$. Thus

$$
\phi(y) = \lim \phi(Tx_n) = \lim \phi(T x_n) = \phi(T x).
$$

So $Tx = y$. It now follows by the closed graph theorem that $T$ is continuous.

It must be emphasised that discontinuous linear operators, defined on incomplete spaces, arise naturally in ZF + DC. For example, there is an abundance of unbounded operators defined on dense subspaces of a Hilbert space. But, for linear operators defined on the whole of a Hilbert space the following theorem holds in ZF + DC + BP.
THEOREM 6. Let $H$ be a Hilbert space and let $T:H \to H$ be a linear operator defined on the whole of $H$. Then $T$ is bounded.

Let $H$ be any Hilbert space. Then, for each nonzero $x$ in $H$, the linear functional $f$, defined by $f(y) = \langle y, x \rangle$, does not vanish at $x$. So $H$ has a separating family of linear functionals.

This implies that, in ZFC, we cannot obtain discontinuous operators on (the whole of) a Hilbert space except by invoking an ‘uncountable’ form of the axiom of choice.

REFERENCES