THE AFFINE STRUCTURES ON THE REAL TWO-TORUS. I

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We wish to complete the study of the affine structures on the real affine 2-tori \(T^2\), following N. H. Kuiper \([2]\), J. P. Benzecri \([1]\) and others.

The category of the affine manifolds is defined, as usual, by the manifolds equipped with maximal atlas whose coordinate transformations are affine transformations \(y^i = \sum a^i_j x^j + b^i, a^i_j, b^i \in \mathbb{R}\), in the cartesian space \(\mathbb{R}^n\), and by the maps which are expressed locally with affine transformations in terms of the affine charts.

Our main result asserts that the affine structures on \(T^2\) are completely determined by the holonomy groups, in which, however, the concept of the holonomy group requires a slight modification as follows.

Given an affine manifold \(M\), its universal covering manifold \(\tilde{M}\) with the induced affine structure is immersed equidimensionally into \(\mathbb{R}^n\) by an affine map \(d\). The map \(d\) gives rise to a homomorphism \(\eta: \pi_1(M) \to A(\mathbb{R}^n)\) of the fundamental group into the affine group \(A(\mathbb{R}^n)\) in such a way that \(d\) is \(\pi_1(M)\)-equivariant with respect to the action of \(\pi_1(M)\) on \(\mathbb{R}^n\) through \(\eta\). The image of \(\eta\) is called the holonomy group \(H\) of \(M\), which is unique up to an inner automorphism of \(A(\mathbb{R}^n)\). Here \(A(M)\), in general, denotes the affine automorphism group of the affine manifold \(M\).

When the image \(d\tilde{M}\) is not simply connected, we switch to its universal covering \((d\tilde{M})^\sim\) from \(\mathbb{R}^n\); that is, we construct an affine immersion: \(d^*: \tilde{M} \to (d\tilde{M})^\sim\) which covers \(d\) and a homomorphism \(\eta^*: \pi_1\tilde{M} \to A((d\tilde{M})^\sim)\) accordingly. Now the modified holonomy group \(H^*\) of \(M\) is by definition the image \(\eta^*(\pi_1\tilde{M})\). When \(d\tilde{M}\) is simply connected, we simply put \(H^* = H\). At any rate \(H^*\) can be regarded as a subgroup of the universal covering group \(A(\mathbb{R}^2)^\sim\) of \(A(\mathbb{R}^2)\).

**Theorem 1.** Two affine structures on \(T^2\) are isomorphic if and only if the modified holonomy groups are conjugate in \(A(\mathbb{R}^2)^\sim\).

The difficulty in the proof lies in establishing that \(d\) is a covering map onto \(d\tilde{M}\). The difficulty may be illustrated by the fact that a surjective immersion of \(\mathbb{R}^2\) onto itself is not always a diffeomorphism. In any case, that \(d\) is a covering implies that \(T^2\) is affine isomorphic with \((d\tilde{M})^\sim/H^*\).

In order to describe the classification of \(H^*\) it is convenient to state the following theorem.

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Theorem 2. For any affine torus \( T^2 \), the affine group \( A(T^2) \) admits nonempty open orbits.

In the transitive case, \( H^* \) is characterized as a lattice subgroup \( \cong \mathbb{Z}^2 \) of a maximal connected abelian subgroup \( G^* \cong \mathbb{R}^2 \) of \( A(R^2) \). The projection \( G = \pi(G^*) \) of \( G^* \) in \( A(R^2) \) is listed below. Since \( G^* \) acts on the affine plane \( R^2 \) almost effectively, \( G^* \) has the induced affine structure, and so \( G^*/H^* \) becomes an affine torus naturally. In the intransitive case, the situation is more complicated; the affine 2-torus \( T^2 \) is then partitioned into several, say \( n \), isomorphic open cylinders and their boundaries (which are closed geodesics in one and the same homotopy class \( \alpha \) in \( \pi_1(T^2) \); those cylinders together constitute the open orbit of \( A(T^2) \)). To be more precise, \( T^2 \) has a cylinder \( R \times S^1 \) as an affine (regular) covering space which admits the affine transformations \( \beta(k):(x, y) \rightarrow (x + k, y) \), \( k \in \mathbb{Z} \), and the covering group is generated by \( \beta(n) \). \( H^* \) is contained in a 2-dimensional abelian subgroup \( G^* \) of \( A(R^2) \) which is saturated (viz. \( G^* = \pi^{-1}(\pi(G^*)) \)) with respect to the projection \( \pi:A(R^2) \rightarrow A(R^2) \) and whose image under \( \pi \) has the identity component \( G \) of type (I-1) or (I-2) in the list below. In particular \( \pi(G^*) \) is a linear transformation group having no translation part. \( \pi(G) \) is generated by \( G \) and the reflection, \( -1 \), with respect to the fixed point of \( G \). \( G^* \) is isomorphic with \( \text{Ker} \pi \times \pi(G^*) \cong \mathbb{Z} \times G \). Now \( H^* \) is generated by two members \( \alpha^*, \beta^* \) such that we have \( \alpha^* = (0, \alpha) \) and \( \beta^* = \beta(n) = (n, \beta) \) in the above correspondence, and that \( \alpha \) is expanding (viz. the eigenvalues of the linear map \( \alpha \) are greater than one and this is a characterization of \( H^* \)).

A question yet to be answered would be: What is the whole picture of all the affine structure of \( T^2 \)? We intend to answer this question in a forthcoming paper.

Finally we list the conjugate classes of the maximal abelian connected subgroups \( G \) of \( A(R^2) \), writing \( (a_{ij}, b_{ij}) \) for the affine transformation \( (x, y) \rightarrow (ax + by + p, cx + dy + q) \). \( G \) consists of

(I-1): \[
\begin{pmatrix}
a & b & 0 \\
0 & a & 0
\end{pmatrix},
\]

(I-2): \[
\begin{pmatrix}
a & 0 & 0 \\
0 & 0 & d
\end{pmatrix},
\]

(I-3): \[
\begin{pmatrix}
u & v & 0 \\
-v & u & 0
\end{pmatrix},
\]

(II): \[
\begin{pmatrix}
1 & 0 & p \\
0 & 0 & d
\end{pmatrix},
\]

(III-1): \[
\begin{pmatrix}
1 & b & p \\
0 & 1 & b
\end{pmatrix},
\]

(III-2): \[
\begin{pmatrix}
1 & 0 & p \\
0 & 1 & q
\end{pmatrix},
\]

(III-3): \[
\begin{pmatrix}
1 & b & p \\
0 & 1 & 0
\end{pmatrix},
\]
where $a > 0$, $d > 0$, $(u, v) \neq (0, 0)$ and the others are arbitrary real numbers.

REFERENCES