CONTRACTING EXTENSIONS AND CONTRACTIBLE GROUPS

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Wiener's classical tauberian theorem has been extended recently to some noncommutative, noncompact groups (see [1], [3], [8] and [10]). Our Theorems 1 and 2 are Wiener type theorems, and interest in them led to the study of contractible groups. It was rather surprising that all contractible Lie-groups are unipotent matrix groups (Theorem 3).

1. Contracting group extensions. A locally compact group \( N \) is contractible provided it has enough contractions, i.e., for any compact set \( K \subset N \) and any neighborhood \( W \) of the identity in \( N \), there is a homeomorphic automorphism \( h \in \text{Aut} \ N \) such that \( hK \subset W \). The ordered pairs \((K, W)\) form a directed set with respect to the relation \( \preceq \), defined by \((K, W) \preceq (K', W')\) if and only if \( K \subseteq K' \) and \( W \supseteq W' \). For every \( n = (K, W) \) choose a contraction \( h_n \) with \( h_nK \subset W \), then \( \{h_n\} \) is a net and for any compact set \( K \subset N \) we have \( \lim_n h_nK = \{e\} \) (\( e \) the neutral element of \( N \)).

A locally compact group \( G \) is a contracting extension of its normal subgroup \( N \) provided the set of restrictions to \( N \) of inner automorphisms of \( G \) contains enough contractions of \( N \). Thus \( N \) must be contractible to admit contracting extensions. For example, if \( G^* \subseteq \text{Aut} \ N \) is a locally compact group and contains enough contractions of \( N \), then the semi-direct product \( G = G^* \rtimes N \) is a contracting extension of \( N \).

If \( G \) is an extension of \( N \) and \( G = G/N \) is the corresponding factor group we will usually denote their elements respectively by \( x, \xi, \hat{x} \), their (left) Haar measures by \( dx, d\xi, d\hat{x} \), and their moduli by \( \Delta, \delta \) and \( \Delta^* \). We suppose that Weil's formula \( dx = d\xi \ d\hat{x} \) holds.

Let us suppose for a moment that \( G \) is separable (i.e. has a countable basis of open sets). Then there exists a measurable cross-section \( \sigma \) of \( G \) with respect to \( N \) (cf. [9]); i.e., there is a measurable function \( \sigma : G^* \to G \) with \( \sigma(\hat{x}) \in \hat{x} = xN \) and \( \sigma(e) = e \). Suppose further that there is a net \( \{h_n\} \) of contractions of \( N \) as above, such that \( \lim_n h_n(x) \) exists for locally almost


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all }x\text{ in }G\text{ (with respect to }dx).\text{ Let }\sigma_{n}(\hat{x}) = h_{n}(\sigma(\hat{x})).\sigma_{n}\text{ is then a measurable function and }\rho(\hat{x}) = \lim_{n} \sigma_{n}(\hat{x})\text{ exists locally almost everywhere on }G.\text{ Since }G\text{ is separable it is metrizable, and the net }\{h_{n}\}\text{ can be replaced by a sequence. By Egoroff's theorem (cf. [2]) we have the property:}

(E) There is a measurable cross-section }\sigma\text{ of }G\text{ with respect to }N\text{ and a measurable function }\rho\text{ from }G\text{ into }G;\text{ and for each compact set }K < G\text{ and every }\varepsilon > 0,\text{ there is a compact set }K_{1} < K\text{ such that }d\hat{x}(K\setminus K_{1}) < \varepsilon\text{ and the restrictions }\sigma_{n}\big|_{K_{1}}\text{ are continuous and converge uniformly to }\rho\text{ as functions on }K_{1}.

From now on we will not use the separability of }G\text{ but we will suppose that the property (E) holds.

Let }L^{1}(G)\text{ be the set of all Haar-measurable and absolutely summable complex-valued functions on }G.\text{ With the usual convolution and involution, }L^{1}(G)\text{ is an involutive Banach algebra, and }L^{\infty}(G)\text{ is its Banach space dual. }G\text{ acts weak-* continuously on }L^{\infty}(G)\text{ by the usual left and right translations. Subspaces which are closed under these actions are called bi-invariant. }

An involutive Banach algebra }B\text{ is said to have the Wiener property if and only if:}

(W) Every proper closed two-sided ideal }I \triangle B\text{ is contained in the kernel of an irreducible, continuous *-representation of }B\text{ on some Hilbert space.}

}B\text{ is said to be tauberian if and only if it has the property:}

(T) Every proper, closed two-sided ideal }I \triangle B\text{ is contained in a maximal modular two-sided ideal of }B.

We will say that a group }G\text{ is tauberian (or has property (W)) if }L^{1}(G)\text{ is tauberian (or satisfies (W)).}

THEOREM 1. Let }G\text{ be a contracting extension of }N\text{ satisfying (E). If }G/N\text{ satisfies (W) or (T), then so does }G.

The proof of this theorem is based on the following lemma and proposition. Since the canonical projection }p:G \rightarrow G\text{ is continuous and open and the function }\rho\text{ is measurable, the composite map }r = \rho \circ p:G \rightarrow G\text{ is measurable.

LEMMA 1. Let }G\text{ be a contracting extension of }N\text{ satisfying (E), and let }M\text{ be a weak-* closed, bi-invariant subspace of }L^{\infty}(G).\text{ If }\phi \in M\text{ is left uniformly continuous on }G\text{ then }\phi \circ r \in M.

PROPOSITION 1. Let }G\text{ and }M\text{ be as in Lemma 1, and let }M_{0}\text{ be the subset of all }\phi \in M\text{ which are constant on the cosets with respect to }N.\text{ Then }M_{0}\text{ is a nontrivial, bi-invariant subspace of }M.

PROPOSITION 1'(DUAL VERSION). Let }G\text{ be as above. If }I\text{ is a proper,
closed two-sided ideal in $L^1(G)$, and if $J$ is the kernel of the morphism $f \to f^\ast$ of $L^1(G)$ onto $L^1(G)$ (where $f^\ast(x) = \int_N f(x\xi)\,d\xi$), then the closure $\text{cl}(I + J)$ is a proper, closed, two-sided ideal in $L^1(G)$; equivalently the closure $\text{cl}(I\cdot)$ of the image of $I$ under the above morphism is a proper closed two-sided ideal in $L^1(G)$.

2. Some extensions of contractible algebras. Let $A$ be an involutive Banach algebra on which a locally compact group $G$ acts strongly continuously by isometric, involutive, algebra automorphisms $T_x$, $x \in G$. The algebra $A$ is $T$-contractible provided that there is a net $\{x_n\}$ in $G$ such that

(i) $\lim_n (T_{x_n}a)b$ exists in $A$ for all $a, b \in A$, and

(ii) for some $u \in A$ the net $\{T_{x_n}u\}$ is an approximating unit for $A$.

For example, if $N$ is a contractible group, $A = L^1(N)$ and $G = \text{Aut } N$ is a locally compact group, then $A$ is $T$-contractible if we define $T$ by $T_x(f)(\xi) = \Delta(x^{-1}) \cdot f(x^{-1}\xi)$ for $x \in G$, $f \in A$, $\xi \in N$. In fact $T_{x_n}f$ converges to the scalar $\hat{a}(f) = \int_N f(\xi)\,d\xi$. Since $A$ contains approximating units, $A$ can be isometrically imbedded in its adjoint algebra $A^b$, which is itself an involutive Banach algebra with unit (cf. [7, §3]).

**Lemma 2.** Let $A$ be a $T$-contractible algebra. The equation

$$R_a b = \lim_n (T_{x_n}a)b \quad (a, b \in A)$$

defines an involutive representation $R$ of $A$ into its adjoint algebra $A^b$. The kernel $j = \ker R$ of $R$ is $G$-invariant, if the $x_n$ belong to the center of $G$.

Let $L = L(G, A; T)$ be the generalized $L^1$-algebra with trivial factor system (cf. [7, §1]). As a Banach space, $L$ is isomorphic to the projective tensor product $L^1(G) \otimes A$. The convolution of $f, g \in L$ is defined by the Bochner integral $f \ast g(x) = \int T_x f(xy) \cdot g(y^{-1})\,dy$, and the involution by $f^\ast(x) = \Delta(x^{-1}) T_{x^{-1}}f(x^{-1})^\ast$. $L$ can be viewed as an extension of the algebra $A$ by the group $G$ (cf. [4]).

Suppose $j = \ker R$ is $G$-invariant. Let $A' = A/j$ be the involutive Banach algebra quotient of $A$ by $j$, and define $T'$ on $A'$ by $T_x(a + j) = (T_xa) + j$. The canonical projection $A \to A'$ induces an isometric isomorphism $L/J \cong L = L(G, A'; T)$ which we denote (par abuse) by $R^\ast$ (cf. [7, §5]). The kernel $J$ of $R^\ast$ can be identified with $L_1(G) \otimes j$.

**Lemma 3.** Let $A$ be a $T$-contractible algebra and assume that $j = \ker R$ is $G$-invariant. Let $J = \ker R^\ast$ be as above.

(i) $\lim_n (T_{x_n}f)^\ast g = 0$ for all $f \in J$ and $g \in L$, where $(T_xf)(y) = T_y(f(y))$.

(ii) Let $p_i$ be an approximating unit of $L^1(G)$; if $R_u = \text{id}_A$ for some $u \in A$ and $p_{i,n} = T_{x_n}(p_i \otimes u) = p_i \otimes T_{x_n}u$, then $\{p_{i,n}\}$ is an approximating unit of $L$, where $(i, n) \geq (i', n')$ iff $i \geq i'$ and $n \geq n'$.
Proposition 2. Let $A$ be a $T$-contractible algebra and assume that $j = \ker R$ is $G$-invariant. If $I$ is a proper, closed, two-sided ideal in $L = L(G, A; T)$ then so is the closure of $I + J$.

By Proposition 2, $L$ will be wienerian (W) or tauberian (T) if $L$ has the respective property.

Theorem 2. Let $A$ be a $T$-contractible algebra. Let $R$ be as in Lemma 1, but assume that each $R_a$ is a scalar multiple of the identity operator. Assume that $j = \ker R$ is $G$-invariant. If $G$ satisfies (W) or (T) then so does $L = L(G, A; T)$.

The method of proof in this paragraph is essentially the same as in [10], whereas the method in §1 is new, and different from the method in [3].

3. Contractible Lie groups and Lie algebras. A few facts about contractible groups in general are collected in

Proposition 3. Let $G$ be a nontrivial contractible group.

(i) $G$ is neither compact nor discrete.

(ii) If $G$ is locally connected, then also globally.

(iii) If $G$ is locally simply connected, then also globally.

(iv) If $G$ has a nontrivial compact subgroup, then it has arbitrarily small ones.

(v) If $G$ has a compact open subset, then $G$ is totally disconnected.

Let $K$ be a nondiscrete, complete field of characteristic 0, and let $\lambda \to |\lambda|$ be a norm (= valuation) of $K$. Since $K$ is nondiscrete there are nonzero $\lambda_n \in K$ with $\lim_n |\lambda_n| = 0$. If $M \subset K$ is (norm-) bounded then the diameters of the sets $\lambda_nM$ converge to 0. Multiplication by a scalar $\lambda_n \neq 0$, defines an automorphism of $K$'s additive group. The additive group of $K$ is thus contractible if locally compact.

Let $\mathcal{G}$ be a finite-dimensional Lie algebra over $K$ with Lie product $(x, y) \to [x, y]$ and norm $x \to |x|$ for which $|[x, y]| \leq |x| \cdot |y|$. The norm $|h|$ of a Lie homomorphism $h$ of $\mathcal{G}$ is the norm of $h$ as a linear operator of the normed space $\mathcal{G}; |h| = \sup\{|hx|; |x| \leq 1\}$.

A contraction of the Lie algebra $\mathcal{G}$ is a Lie automorphism $h$ with $|h| < 1$. If $\mathcal{G}$ has one contraction $h$, then it has enough contractions and we call $\mathcal{G}$ contractible: the powers $h^n$ of $h$ map every bounded set eventually into any 0-neighborhood of $\mathcal{G}$, because their norms $|h^n|$ converge to 0.

Proposition 4. Finite dimensional contractible Lie algebras over $K$ are nilpotent.

Examples. (1) All freely generated, nilpotent Lie algebras are contractible.
(2) All nilpotent Lie algebras of dimension $\leq 6$ are contractible, but some are not freely generated. (This last result is based on the classification of these Lie algebras in [11].)

A unipotent matrix over $K$ is an (upper) triangular matrix of finite order with coefficients from $K$ and 1's in the main diagonal. A unipotent group over $K$ is (up to a global isomorphism) a group of unipotent matrices with matrix multiplication as its group operation, which is complete with respect to a norm topology on the respective matrix ring. The topology of a unipotent group does not depend on the choice of norm because $K$ (etc.) is completely metrizable, and Baire's theorem applies.

**Proposition 5.** If $\mathfrak{g}$ is a finite dimensional nilpotent Lie algebra over $K$ (not necessarily contractible) then $\mathfrak{g}$ can be imbedded into an associative matrix algebra $A$ over $K$, such that the power series $\exp(x) = \sum_{n \geq 0} x^n/n!$, as evaluated in $A$, reduces to a polynomial for all $x \in \mathfrak{g}$, and such that the global image $\exp^\mathfrak{g}$ of $\mathfrak{g}$ under $\exp$ is a unipotent group.

The proof of this proposition depends on the theorems of Ado, Lie and Campbell-Hausdorff (cf. e.g. [5]).

**Theorem 3.** If $G$ is a contractible Lie group of finite dimension over the field $\mathbb{R}$ of real numbers or the field $\mathbb{Q}_p$ of $p$-adic numbers, then $G$ is a unipotent group.

In the real case, the proof of Theorem 3 is achieved through Propositions 6 and 7 below, which in turn depend on classical theorems. In the $p$-adic case, however, we rely on results from [6], notably the "inversion of the Campbell-Hausdorff formula" [ibid., IV, 3.2.3].

**Proposition 6.** The Lie algebra $\mathfrak{g}$ of a contractible Lie group $G$ over $\mathbb{R}$ is contractible and thus nilpotent.

**Proposition 7.** If $G$ is a connected and simply connected nilpotent Lie group over $\mathbb{R}$ (not necessarily contractible), then $G$ is a unipotent group.

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