A STRUCTURE THEORY OF JORDAN PAIRS

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1. Definitions. Let $V^{\sigma}$ be modules over a ring $k$ where $\sigma = \pm$. For a quadratic map $Q^{\sigma}: V^{\sigma} \to \text{Hom}_k(V^{-\sigma}, V^{\sigma})$ let

$$L'(x, y)z = Q^{\sigma}(x + z)y - Q^{\sigma}(x)y - Q^{\sigma}(z)y = \{xyz\}.$$ 

A Jordan pair over $k$ is a pair $(+, -)$ of $k$-modules together with a pair $(+, -)$ of quadratic maps $Q^{\sigma}: V^{\sigma} \to \text{Hom}_k(V^{-\sigma}, V^{\sigma})$ such that the identities

\begin{align*}
(JP1) & \quad L'(x, y)Q^{\sigma}(x) = Q^{\sigma}(x)L'^{-\sigma}(y, x), \\
(JP2) & \quad L'(Q^{\sigma}(x)y, y) = L'(x, Q'^{-\sigma}(y)x), \\
(JP3) & \quad Q'^{-\sigma}(Q^{\sigma}(x)y) = Q^{\sigma}(x)Q'^{-\sigma}(y)Q^{\sigma}(x),
\end{align*}

hold in all base ring extensions. Jordan pairs have first been studied by K. Meyberg in [6], although not in the present form.

A homomorphism $h: \mathcal{V} \to \mathcal{W}$ of Jordan pairs is a pair $h = (h^+, h^-)$ of $k$-linear maps $h^\sigma: V^\sigma \to W^\sigma$ such that $h^\sigma Q^\sigma(x) = Q^\sigma(h^\sigma(x))h^{-\sigma}$, for all $x \in V^\sigma$. The opposite of $\mathcal{V}$ is $\mathcal{V}^{\text{op}} = (V^-, V^+)$ with quadratic maps $(Q^-, Q^+)$. An antihomomorphism from $\mathcal{V}$ to $\mathcal{W}$ is a homomorphism from $\mathcal{V}^{\text{op}}$ to $\mathcal{W}^{\text{op}}$. An antiautomorphism $\eta$ of $\mathcal{V}$ is called an involution if $\eta^{-\sigma} \eta^\sigma$ is the identity on $V^\sigma$.

2. Connections with Jordan algebras and Jordan triple systems. There is a one-to-one correspondence between Jordan triple systems (cf. [7]) and Jordan pairs with involution as follows: If $\eta$ is an involution of the Jordan pair $\mathcal{V}$ then $V^+$ becomes a Jordan triple system with quadratic operators $P(x) = Q^+(x)\eta^+$. If conversely $(V, P)$ is a Jordan triple system then $(V, V)$ is a Jordan pair with $Q^+(x)y = P(x)y$ and involution $\eta^\sigma = \text{Id}_V$. The structure group of the Jordan triple system is the automorphism group of the corresponding Jordan pair.

Let $\mathcal{V}$ be a Jordan pair. An element $a \in V^+$ is called invertible if $Q^+(a)$ is invertible. There is a one-to-one correspondence between Jordan pairs containing invertible elements and isotopism classes of unital quadratic operators.


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Jordan algebras: if \( a \in V^+ \) is invertible then \( \mathcal{J} = (V^+, U(a), a) \) where \( U(a)(x) = Q^+(x)Q^+(a)^{-1} \) is a unital quadratic Jordan algebra. If \( b \) is also invertible then \( U(b)(x) = U(a)(x)U(a)(b)^{-1} \) and therefore \( (V^+, U(b), b) \) is an isotope of \( \mathcal{J} \). Conversely, if \( (J, U, e) \) is a unital quadratic Jordan algebra then \( (J, U) \) is a Jordan triple system, and the corresponding Jordan pair contains invertible elements. Again the structure group of the Jordan algebra is the automorphism group of the corresponding Jordan pair.

3. The radical. Let \( V^\sigma \) be a Jordan pair and let \( y \in V^{-\sigma} \). The k-module \( V^\sigma \) becomes a quadratic Jordan algebra, denoted \( V^\sigma_\sigma \), with quadratic operators \( U(x) = Q^\sigma(x)Q^{-\sigma}(y) \) and squaring \( x^2 = Q^\sigma(x)y \). A pair \( (x, y) \in V^\sigma \times V^{-\sigma} \) is called quasi-invertible if \( x \) is quasi-invertible in \( V^\sigma_\sigma \). The Jacobson radical of \( V^\sigma \) is \( \text{Rad} V^\sigma = (\text{Rad} V^+, \text{Rad} V^-) \) where \( \text{Rad} V^\sigma = \{ x \in V^\sigma : (x, y) \text{ quasi-invertible for all } y \in V^{-\sigma} \} \). It has the same properties as the radical of a Jordan algebra (cf. [2]).

4. Peirce decomposition. An element \( x \in V^\sigma \) is called (von Neumann-) regular if \( x = Q^\sigma(x)y \) for some \( y \in V^{-\sigma} \). An idempotent of \( V^\sigma \) is a pair \( c = (c^+, c^-) \in V^+ \times V^- \) such that \( c^\sigma = Q^\sigma(c^\sigma)c^- \), i.e., \( c^\sigma \) is an idempotent in \( V^\sigma_{c^-\sigma} \).

Lemma. If \( c^+ \in V^+ \) is regular then there exists \( c^- \in V^- \) such that \( (c^+, c^-) \) is an idempotent.

Indeed, if \( c^+ = Q^+(c^+)y \) set \( c^- = Q^-(y)c^+ \). Then it follows from (JP3) that \( (c^+, c^-) \) is an idempotent. For an idempotent \( c \) we have the Peirce decomposition \( V^\sigma = V^\sigma_1(c) \oplus V^\sigma_2(c) \oplus V^\sigma_0(c) \) where \( V^\sigma_1(c) = (V^+_1, V^-_1) \) is a subpair of \( V^\sigma \) and \( V^\sigma_i \) is the Peirce-i-space of the Jordan algebra \( V^\sigma_{c^-\sigma} \). Also \( V^\sigma_1 \) is a unital quadratic Jordan algebra (as subalgebra of \( V^\sigma_{c^-\sigma} \)) with unit element \( c^\sigma \). Two idempotents \( c \) and \( d \) are called orthogonal if \( d \in V^\sigma_0(c) \). As for Jordan algebras there is a Peirce decomposition with respect to a system of pairwise orthogonal idempotents.

5. Inner ideals. A submodule \( I \) of \( V^\sigma \) is called an inner ideal if \( Q^\sigma(I)V^{-\sigma} = I \). An element \( x \in V^\sigma \) is called an absolute zero divisor if \( Q^\sigma(x) = 0 \). Let \( 0 \neq I \subseteq V^+ \) be a minimal inner ideal. Then either \( I = k \cdot x \) where \( x \) is an absolute zero divisor, or \( I = Q^+(x)V^- \) for all \( 0 \neq x \in I \); in particular, \( I \) consists of regular elements. In the latter case, let \( 0 \neq c^+ \in I \) and choose \( c^- \in V^- \) such that \( (c^+, c^-) \) is an idempotent which is possible by the Lemma. Then \( I = V^+_1 \) is a Jordan division algebra. This shows that there are no minimal inner ideals of type II (cf. [2]) for Jordan pairs.

6. Artinian Jordan pairs. The principal inner ideal generated by an element \( x \in V^\sigma \) is \( Q^\sigma(x)V^{-\sigma} \). A Jordan pair \( V^\sigma \) is called Artinian if \( V^\sigma \) satisfies the descending chain condition for principal inner ideals.
Theorem 1. Let \( \mathcal{V} \) be Artinian. Then there exists an idempotent \( c \) such that \( \mathcal{V}_0(c) \subseteq \text{Rad} \mathcal{V} \) and \( c = c_1 + \cdots + c_n \) is the sum of orthogonal primitive idempotents. If \( \text{Rad} \mathcal{V} = 0 \) then \( c \) is maximal (i.e., \( \mathcal{V}_0(c) = 0 \)) and the \( c_i \) are completely primitive.

Here an idempotent \( c \) is called primitive if \( c^2 = c \) is the only idempotent in \( V_1^2 \), and completely primitive if \( V_1^2 \) is a Jordan division algebra. Thus in contrast to Jordan algebras, a Jordan pair always has capacity.

For \( \mathcal{V} \) Artinian the following conditions are equivalent:
(i) \( \mathcal{V} \) is regular; i.e., \( V^a \) consists of regular elements,
(ii) \( \text{Rad} \mathcal{V} = 0 \).
(iii) \( V^a \) contains no absolute zero divisors \( \neq 0 \).

Theorem 2. The regular Artinian Jordan pairs are precisely the finite direct products of simple regular Artinian Jordan pairs.

Conversely we conjecture: A simple Artinian Jordan pair is regular. This has been proved in the finite-dimensional case.

7. Alternative pairs. A pair \( \mathcal{A} = (A^+, A^-) \) of \( k \)-modules together with trilinear maps \( A^a \times A^- \times A^a \to A^a \), \( (x, y, z) \mapsto \langle xyz \rangle \), is called an alternative pair over \( k \) if the identities
\[
\begin{align*}
\text{(AP1)} & \quad \langle uv \langle xyz \rangle \rangle + \langle xy \langle wz \rangle \rangle = \langle \langle uvx \rangle yz \rangle + \langle x \langle vuy \rangle z \rangle, \\
\text{(AP2)} & \quad \langle xy \langle xyz \rangle \rangle = \langle \langle xyx \rangle yz \rangle, \\
\text{(AP3)} & \quad \langle uv \langle xyx \rangle \rangle = \langle \langle uvx \rangle yx \rangle,
\end{align*}
\]
hold for all \( x, u, z \in A^a, y, v \in A^- \). We say \( \mathcal{A} \) is associative if the identity
\[
\langle uv \langle xyz \rangle \rangle = \langle u \langle vxy \rangle z \rangle = \langle \langle uvx \rangle yz \rangle
\]
is satisfied. The relation between alternative pairs and Jordan pairs is as follows: If \( c \) is a maximal idempotent of the Jordan pair \( \mathcal{V} \) then \( \mathcal{V}_{1/2}(c) = (V_{1/2}^a, V_{1/2}^{-a}) \) is an alternative pair with \( \langle xyz \rangle = \{ [xyc]^a \} e^{-a} z \}. \) Conversely, every alternative pair \( \mathcal{A} \) may be realized as the Peirce-\( 1/2 \)-space of a Jordan pair \( \mathcal{V} = \mathcal{V}(\mathcal{A}) \) with respect to a maximal idempotent. The construction of \( \mathcal{V}(\mathcal{A}) \) is similar to the one for alternative triple systems (cf. [4], [7]).

Every alternative pair \( \mathcal{A} \) gives rise to a Jordan pair \( \mathcal{A}^J \) having the same underlying modules and quadratic maps \( Q^a(x)y = \langle xyx \rangle \). The concepts of radical, idempotent, (principal) inner ideal, and Artinian are the same for \( \mathcal{A} \) and \( \mathcal{A}^J \).

8. Classification. Let \( \mathcal{V} \) be a regular simple Artinian Jordan pair over \( k \). Since these properties are invariant under restriction of the ring of scalars we may assume \( k = \mathbb{Z} \). Let \( \mathcal{V} = \mathcal{V}_{1/2}(c) \oplus \mathcal{V}_{1/2}(c) \) with respect to a maximal idempotent. If \( \mathcal{V}_{1/2}(c) = 0 \) then \( \mathcal{V} \) contains invertible elements
and is therefore essentially a unital quadratic Jordan algebra. The structure of these algebras is well known ([1], [2], [5]). If \( \mathcal{A} = \mathcal{V}_{1/4}(c) \neq 0 \) then \( \mathcal{A} \) is a simple regular Artinian alternative pair, and \( \mathcal{V} = \mathcal{V}(\mathcal{A}) \). For this reason we first classify alternative pairs.

**Theorem 3.** The simple regular Artinian alternative pairs are up to isomorphism precisely the following.

(Ia) \((M^+, M^-)\) where \( M^a \) is a left module over a simple associative Artinian ring \( R^a \), with product \( (xyz) = q^a(x, y)z \). Here \( R^{-a} \) is anti-isomorphic with \( R^a \) under an anti-isomorphism \( a \mapsto a \), and \( q^a : M^a \times M^{-a} \rightarrow R^a \) is sesquilinear and nondegenerate, i.e., it is \( R^a \)-linear in the first variable, \( q^a(x, y) = q^{-a}(y, x) \), and \( q^a(x, M^{-a}) = 0 \) implies \( x = 0 \).

(II) \((C^+, C^-)\) where \( C^a = C^+ \) is a Cayley algebra, \( C^- = C^{op} \), and \( (xyz) = (xy)z \). Here \( x \mapsto \bar{x} \) denotes the canonical anti-isomorphism \( C^a \rightarrow C^{-a} \).

(III) \((A^+, A^-)\) where \( A^a \) is a vector space over a field \( K \), with product \( (xyz) = \sigma(z, I^a(y)) \cdot x + I^a(y) \sigma(x, z) \). Here \( \sigma \) is a nondegenerate alternating bilinear form on \( A^a \), and \( I^a : A^{-a} \rightarrow A^a \) is a \( K \)-linear map leaving \( \sigma \) invariant and satisfying \( I^a \circ I^{-a} = -Id \).

The pairs of type (I) are associative, the others are properly alternative.

From this we compute the associated Jordan pairs \( \mathcal{V} = \mathcal{V}(\mathcal{A}) \). For \( \mathcal{A} \) of type (Ib) with \( M^a \) not finitely generated as \( R^a \)-module and of type (III) with \( A^a \) infinite dimensional \( \mathcal{V}(\mathcal{A}) \) is not simple. Using the well-known classification of simple Artinian Jordan algebras we have the following result (where convenient, Jordan algebras or triple systems are listed instead of the corresponding Jordan pair).

**Theorem 4.** The simple regular Artinian Jordan pairs are up to isomorphism precisely the following.

(0) Jordan division algebras.

(I) \( \mathcal{A}^J \) where \( \mathcal{A} \) is an alternative pair of type (Ia).

(II) Alternating \((n \times n)\)-matrices over a field, \( n \geq 2 \).

(III) Outer ideals containing 1 in Jordan algebras of hermitian \((n \times n)\)-matrices over an associative division algebra with involution, \( n \geq 2 \).

(IV) Outer ideals containing 1 in Jordan algebras of nondegenerate quadratic forms over a field.

(V) \((M_{1,2}(C), M_{1,2}(C^{op}))\), \((1 \times 2)\)-matrices over a Cayley algebra \( C \), resp. \( C^{op} \), with \( Q^a(xy) = x(I^a y) \). Here \( I^a y \) denotes the transpose of \( y \) with coefficients in the opposite algebra.

(VI) Hermitian \((3 \times 3)\)-matrices over a Cayley algebra.

Details will appear elsewhere.
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