COHOMOLOGY AND WEIGHT SYSTEMS FOR NILPOTENT LIE ALGEBRAS

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Communicated by Dock Rim, July 17, 1973

1. This paper announces results concerning the cohomology groups $H^*(N, N)^T$ where $N$ is in a certain class of finite-dimensional nilpotent Lie algebras over a field $k$ and $T$ is an abelian Lie algebra faithfully represented as a maximal diagonalizable algebra of derivations of $N$; we shall refer to such an $N$ as a $T$-algebra. The additional hypotheses to be placed on the pair $N$, $T$ are inspired by the case when $T$ is a Cartan subalgebra and $T+N=B$ is a Borel subalgebra of a complex semisimple Lie algebra. In that case Kostant has shown [2] that $H^i(N, N)^T=0$ for $i \geq 2$ and the authors applied this result in [3] to conclude that $H^*(B, B)=0$. (A similar argument shows $H^*(P, P)=0$ for $P$ parabolic.) Here we are concerned with the relations between the vanishing of $H^i(N, N)^T$, especially for $i=2$, and the structure of the algebras $N$.

Let $W$ denote the set of weights of $T$ in $N$. If $\dim(T) = \dim(N/N^2) = m$ then the subset of $W$ arising from the induced representation of $T$ on $N/N^2$ has precisely $m$ elements, say $\{\alpha_1, \cdots, \alpha_m\}$. Every $\alpha \in W$ then has a unique representation $\alpha = \sum c_i \alpha_i$ with each $c_i$ a nonnegative integer and $c_i < p$ if the characteristic of $k$ is $p>0$. For such an $\alpha$ we call the sum (in $Z$) $\sum c_i$ the height of $\alpha$ and denote it by $|\alpha|$. For $\alpha$ in $W$, denote by $N_\alpha$ the weight space for $\alpha$ in $N$.

**DEFINITION.** A $T$-algebra is called positive if

(i) $\dim(T) = \dim(N/N^3)$,

(ii) $N$ is graded by the heights of the weights, i.e., if $N(j) = \bigoplus_{|\alpha|=j} N_\alpha$ then $[N(j), N(k)] \subset N(j+k)$.

**Remark.** Condition (ii) is superfluous in characteristic 0. However, in characteristic $p>0$ it has such consequences as $N^r = 0$ for $r > (p-1) \cdot \dim(T)$.

2. When $T$ is a Cartan subalgebra of a complex semisimple Lie algebra $G$, $T+N$ a Borel subalgebra of $G$ and $W$ the weights of $T$ in $N$, it is well known that $N$ is the unique positive $T$-algebra with corresponding weight system $W$. This fact is a special case of the following theorem.

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Key words and phrases. Lie algebra cohomology, weight systems, deformations of Lie algebras.
THEOREM 1. Let $N$ be a positive $T$-algebra and suppose that $H^2(N|N^r, N/N^r)^T = 0$ for $r \geq 4$. If $N'$ is any other positive $T$-algebra inducing the same weight system $W$ with the same height function and $\dim(N'_2) \geq \dim(N_2)$ for each $\alpha \in W$, then $N'$ is necessarily $T$-isomorphic to $N$.

Although the task of verifying the $H^2$ hypothesis in Theorem 1 appears to be quite formidable, we are in fact able to check this condition for a large class of algebras by means of simple observations about $W$. For example, the collection of algebras described in [3, §5], which include the maximal nilpotent ideals of Borel subalgebras, satisfies the assumption. Also, we can show that the collection includes all the positive $T$-algebras $N$ in characteristic $\neq 2$ which satisfy the following:

- If $\alpha_1, \ldots, \alpha_m$ are the elements in $W$ of height 1, then
  1. Every element in $W$ is of the form $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_s}$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq m$.
  2. If $\gamma, \gamma - \alpha_i, \gamma - \alpha_j \in W$ with $i \neq j$, $|\gamma| \geq 3$ then $\gamma - \alpha_i - \alpha_j \in W$ and $\alpha_i + \alpha_j \notin W$.

3. The appearance of $H^2(N|N^r, N/N^r)^T$ in Theorem 1 is suggestive of the role that $H^2(N, N)^T$ plays in a rigidity theorem for algebras acted upon by $T$. The set of $T$-multiplications in a vector space $N$ on which $T$ operates clearly form an algebraic subvariety of the affine space $N^* \wedge N^* \otimes N$. Thus we are led to a restricted deformation theory for such algebras. If $T$ operates diagonally on $N$ then $H^*(N, N)^T$ plays the role in this theory that $H^*(L, L)$ plays in the deformation theory of ordinary Lie algebras (cf. [4]). For example, we have the concept of $T$-rigidity and the theorem: If $\mu$ is a $T$-multiplication on $N$ and if $H^2((N, \mu), (N, \mu))^T = 0$ then $(N, \mu)$ is $T$-rigid. Also, $H^3((N, \mu), (N, \mu))^T$ arises as obstructions to integrability of 2-cocycles.

We remark that the hypothesis $H^2(N|N^r, N/N^r)^T = 0$, for $r \geq 4$, of Theorem 1 is strictly stronger for positive $T$-algebras than the vanishing of $H^2(N, N)^T$.

4. Suppose characteristic $(k) \neq 2$. A remarkable class of positive $T$-algebras over $k$ is obtained by generalizing the Coxeter-Dynkin diagram for $A_i$.

Let $\Gamma$ be an undirected graph [5] with at most one edge connecting any two vertices and without loops (i.e., for any vertex $v$, $(v, v)$ is not an edge). A section graph $g$ of $\Gamma$ is a subgraph such that any edge in $\Gamma$ which connects two vertices of $g$ is in $g$. By a subtree $t$ of $\Gamma$ we shall mean a connected section graph with no circuits.

DEFINITION. A collection $S$ of subtrees of $\Gamma$ will be called admissible if

(i) whenever $t \in S$ and $t'$ is a subtree of $t$ then $t' \in S$. 

(ii) $\Gamma = \bigcup_{t \in S} t$. (So each vertex and each edge in $\Gamma$ is in some $t$ of the collection.)

To each admissible collection $S$ of subtrees of $\Gamma$ we associate a Lie algebra $N_S$ as follows:

Arbitrarily assign a direction $v_i \rightarrow v_j$ to each edge $(v_i, v_j)$ in $\Gamma$. We define $\varepsilon : S \times S \rightarrow \{\pm 1, 0\}$ such that $\varepsilon_{t, t'} = 0$ if either $t \cap t' \neq \emptyset$ or $t \cup t' \notin S$, otherwise $\varepsilon_{t, t'} = 1$ if the unique edge connecting $t$ to $t'$ is directed $t \rightarrow t'$ and $\varepsilon_{t, t'} = -1$ if this edge is directed $t' \rightarrow t$. Now let $N_S$ be the $k$-algebra with basis $S$ such that $[t, t'] = \varepsilon_{t, t'} t t'$. The Jacobi identity is readily verified and so $N_S$ is a nilpotent Lie algebra generated by the set of vertices. Next we point out that, for any vertex $v$ of $\Gamma$, there is a unique derivation $a_v$ of $N_S$ such that $a_v(v) = v$ and $a_v(w) = 0$ for any vertex $w \neq v$. The set of derivations $a_v$ span a maximal diagonalizable algebra $T_S$ of derivations of $N_S$. One sees that $N_S$ is a positive $T_S$-algebra and it is easy to verify that it satisfies property (*). Thus, by Theorem 1, $N_S$ is the unique $T_S$-algebra producing its system of weights. In particular, its isomorphism class is independent of the choice of directions in $\Gamma$.

The simplest of graphs consists of a single path and if, in this case, $S$ is taken to be the collection of all subtrees, $N_S$ is the nilpotent subalgebra corresponding to the positive weights in an algebra of type $A_1$.

Thus, it is not surprising that the algebras $N_S$ have structural and cohomological properties like those of the maximal nilpotent subalgebras of Borel subalgebras. For example, using the classical spectral sequences of Hochschild-Serre [1] as well as combinatorial properties of graphs, we can show:

**Theorem 2.** Let characteristic $(k)=0$. Suppose $\Gamma$ is a graph, $S$ an admissible collection of subtrees. Then $H^i(N_S, N_S)^T_S = 0$ for $i \geq 2$.

Then, following methods used in [3], we prove:

**Theorem 3.** Let characteristic $(k)=0$. If $B_S$ is the semidirect sum $T_S + N_S$ then $H^*(B_S, B_S) = 0$.

Finally, we announce a characterization of the graph algebras by property (*). First we assign to each positive $T$-algebra $N$ a graph $\Gamma(N)$ as follows: Let $\alpha_1, \cdots, \alpha_m$ be the elements in $W$ of height 1, then $\Gamma(N)$ is the graph on $m$ vertices $v_1, \cdots, v_m$ such that $(v_i, v_j)$ is an edge if and only if $\alpha_i + \alpha_j$ is in $W$. Let $S$ be any admissible collection of subtrees of $\Gamma(N)$. Then $T$ acts on $N_S$ via the isomorphism $T \rightarrow T_S$ given by $x \mapsto \sum_i \alpha_i(x)a_{v_i}$. 
THEOREM 4. Let \( N \) be a positive \( T \)-algebra satisfying property \((*)\). For each \( \alpha = \alpha_{i_1} + \cdots + \alpha_{i_r} \) in \( W \), let \( g_{\alpha} \) be the section graph of \( \Gamma(N) \) with vertices \( v_{i_1}, \cdots, v_{i_r} \). Then each such \( g_{\alpha} \) is a subtree of \( \Gamma(N) \); the set \( S = \{ g_{\alpha} | \alpha \in W \} \) is an admissible collection of subtrees of \( \Gamma(N) \); and \( N \) is \( T \)-isomorphic to \( N_S \).

REFERENCES