Let $\gamma(t), -\infty < t < \infty$, be a smooth curve in $\mathbb{R}^n$. For $f$ in $C_0^\infty(\mathbb{R}^n)$ set

$$Tf(x) = \lim_{\varepsilon \to \infty, N \to \infty} \int_{\varepsilon \leq |t| \leq N} \frac{f(x - \gamma(t))}{t} \, dt.$$  

$Tf$ is the Hilbert transform of $f$ along the curve $\gamma(t)$. E. M. Stein [2] raised the following general question: For what values of $p$ and what curves $\gamma(t)$ is $Tf$ a bounded operator in $L^p$? If $\gamma(t)$ is a straight line it is well known that $T$ is bounded for $1 < p < \infty$. Stein and Wainger [3] proved that the operator is bounded for $p = 2$ if

$$\gamma(t) = (|t|^\alpha_1 \text{ sgn } t, \cdots, |t|^\alpha_n \text{ sgn } t), \quad \alpha_i > 0.$$  

Here we show that $Tf$ is a bounded operator in $L^p$ for some $p$ other than 2 and some nontrivial, nonlinear $\gamma$'s. We prove

**Theorem 1.** Let $\gamma(t) = (|t|^\alpha_1 \text{ sgn } t, |t|^\alpha_2 \text{ sgn } t)\alpha_1 > 0, \alpha_2 > 0$. Then $Tf$ is bounded in $L^p$ for $\frac{3}{2} < p < 4$.

**Sketch of the Proof.** The transformation (1) may be expressed as a multiplier transformation. In our case,

$$(Tf)(x, y) = m(x, y)f(x, y)$$

where

$$m(x, y) = \lim_{\varepsilon \to \infty, N \to \infty} \int_{\varepsilon \leq |t| \leq N} \exp\{i |t|^\alpha_1 \text{ sgn } tx + i |t|^\alpha_2 \text{ sgn } ty\} \frac{dt}{t}$$  

($\wedge$ denotes Fourier transform).

By a change of variables we may assume $\alpha_1 = 1$ and $\alpha_2 \geq 1$. Furthermore we may assume $\alpha_2 > 1$, for otherwise we have the case that $\gamma(t)$ is a straight line. Stein [2] showed that $Tf$ is bounded for $p = 2$ if $\gamma(t)$ is a straight line. Stein and Wainger [3] proved that the operator is bounded for $p = 2$ if $\gamma(t)$ is a straight line. Stein and Wainger [3] proved that the operator is bounded for $p = 2$ if $\gamma(t)$ is a straight line. Stein and Wainger [3] proved that the operator is bounded for $p = 2$ if $\gamma(t)$ is a straight line.


1 Partially supported by an NSF grant at the University of Wisconsin.

2 Partially supported by an NSF grant at the University of Minnesota, GP-15832.

Copyright © American Mathematical Society 1974

106
line. Thus in (3) we take $\alpha_1 = 1$ and $\alpha_2 = a > 1$. Clearly $m$ is odd and $m(rx, r^2y) = m(x, y)$, $r > 0$. By using the method of steepest descents and integration by parts we obtain

**Theorem 2.** $m(x, y)$ is infinitely differentiable away from the line $y = 0$.

For $0 \leq |y|/x^a \leq 1$,

$$m(x, y) = m_1(x, y) + m_2(x, y) + m_3(x, y),$$

where, if

$$\lambda = |y|/x^a \quad \text{and} \quad \beta = (a - 1)^{-1},$$

$$m_1(x, y) = \begin{cases} \sum_{j=1}^{n} A_j \lambda^{\beta/2 + \eta_j} \exp(i\lambda^{-\beta} \eta_j), & y \geq 0, \\ 0, & y \leq 0, \end{cases}$$

$$m_2(x, y) = \begin{cases} \sum_{j=1}^{m} B_j \lambda^{\beta/2 + \rho_j} \exp(i\lambda^{-\beta} \xi_j), & y \leq 0, \\ 0, & y \geq 0, \end{cases}$$

$m_3(x, y)$ has continuous second order partial derivatives away from the origin. Here $A_j$ and $B_j$ are complex numbers and $\eta_j$ and $\xi_j$ are real.

We shall consider a multiplier of the form $n(x, y) = g(|y|/x^a)$ where

$$g(\lambda) = \begin{cases} \lambda^{\beta/2} \exp(i\lambda^{-\beta}) \omega(\lambda), & \lambda > 0, \\ 0, & \lambda \leq 0, \end{cases}$$

where $\omega$ is $C^\infty$, has support in $[-1, 1]$ and is identically 1 near $\lambda = 0$. Theorem 2 implies that $m(x, y)$ is a finite sum of multipliers each of which may be treated in the same way as $n(x, y)$. Set

$$g_\beta(\lambda) = \begin{cases} \lambda^{\beta/2} \exp(i\lambda^{-\beta}) \omega(\lambda), & \lambda \geq 0, \\ 0, & \lambda \leq 0, \end{cases}$$

and $n_\beta(x, y) = g_\beta(|y|/x^a)$.

We wish to show

$n_{1/2}$ is a bounded multiplier on $L^p$ for $\frac{4}{3} < p < 4$.

Clearly $n_{\sigma+t\tau}(x, y)$ is a bounded multiplier on $L^2$ (with norm uniformly bounded in $t$). Hence, in view of the interpolation theorem for analytic families of operators, to prove $n_{1/2}$ is a bounded multiplier on $L^p$, $\frac{4}{3} < p < 4$, it suffices to prove

**Theorem 3.** $n_{\sigma+t\tau}$ is a bounded multiplier on $L^p$, $1 < p < \infty$ for $\sigma > 1$, with a bound that is independent of $t$. 
Theorem 3 will in turn follow by arguments similar to Rivière [1], if one can prove the following

**Lemma.** Let \( \psi(r) \) be in \( C^\infty[0, \infty) \) with support in \( [1, 2] \), \( \rho(x, y) = (x^2 + y^2)^{1/2} \), and \( \phi(x, y) = \psi(\rho(x, y)) \). For \( \delta \) positive and small set \( l = \frac{1}{2}(1 + 1/x) + \alpha \) and \( k = (x + 1)/2 \).

Then

\[
\int_{\mathbb{R}^4} (|x|^{2k} + |y|^{2l}) |(n_{\sigma + it}\phi)\gamma(x, y)|^2 \, dx \, dy \leq C
\]

and

\[
\int_{\mathbb{R}^4} (|x|^{2k} + |y|^{2l}) |(n_{\sigma + it}\phi h_{s, u})\gamma(x, y)|^2 \, dx \, dy \leq C[\rho(s, u)]^2.
\]

\( h_{s, u}(x, y) = e^{i(sx + uy)} - 1 \). (\( \gamma \) denotes inverse Fourier transform).

Lemma 2 is proved by (a) proving appropriate analogues of (i) and (ii) if \( k = m + it \), \( m \) a nonnegative integer, \( l = 1 + it \), and \( l = it \), and then (b) using the Phragmén-Lindelöf theorems. Details will appear elsewhere.

**References**


**Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706**

**Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455**