PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

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In this note we announce a result concerning the existence of a periodic solution for a class of periodically perturbed conservative systems. Our result, in a sense, completes a series of investigations originated by W. S. Loud [4]. Also see [1], [2], [3], and [5]. Our techniques are different from those of the authors cited above.

Consider the vector differential equation

\[ x'' + \text{grad } G(x) = p(t) = p(t + 2\pi), \]

where \( p \in C(R, R^n) \), \( G \in C^2(R^n, R) \). This equation can be interpreted as the newtonian equation of a mechanical system subject to conservative internal forces and periodical external forces.

**Theorem 1 (Lazer [1]).** Let \( A \) and \( B \) be real constant symmetric matrices such that if \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \) denote the eigenvalues of \( A \) and \( B \) respectively then there exist integers \( N_k \geq 0 \), \( k = 1, \cdots, n \), such that \( N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2 \).

If, for all \( a \in R^n \), \( A \leq \partial^2 G(a)/\partial x_i \partial x_j \leq B \), then (1) has at most one \( 2\pi \)-periodic solution.

Our theorem establishes the existence part of the preceding theorem. More specifically, we prove

**Theorem 1*. If \( G \), \( A \) and \( B \) satisfy the hypothesis of Theorem 1, then (1) has a \( 2\pi \)-periodic solution.

The key to the proof of our theorem is

**Lemma 1.** Let \( \bar{Q}(t) \) be a real \( n \times n \) symmetric matrix whose elements are bounded, measurable and \( 2\pi \)-periodic on the real line. Let \( A \) and \( B \) be real constant symmetric matrices such that \( A \leq \bar{Q}(t) \leq B \). If \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \cdots \leq \mu_n \) denote the eigenvalues of \( A \) and \( B \) respectively then there

exist integers \( N_k \geq 0, k=1, \ldots, n \), satisfying

\[ N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2. \]

Let \( f(t) \) be a real vector-valued 2\( \pi \)-periodic continuous function with \( \|f(t)\| \leq K, K \) some number. Then there exists a number \( r > 0 \), independent of \( f(t) \), such that for any periodic solution \( u \) of \( u'' + Qu = f \) the inequality \( \|u(t)\|^2 + \|u'(t)\|^2 \leq r^2 \) holds for all \( t \) (we mean \( u' \) absolutely continuous and the preceding equation holds a.e.).

Using this lemma we prove that our theorem follows from a generalization of Poincaré's perturbation theorem (see [3]). The proof of Lemma 1 is too long to give here. A brief sketch may be given along the following line. Assuming that the conclusion of Lemma 1 is false, we construct a sequence of equations of the form

\[ z''_m + Q_m(t)z_m = g_m(t) \quad \text{a.e.} \]

where \( z_m, Q_m \) and \( g_m \) are 2\( \pi \)-periodic (\( Q_m \) symmetric). It is shown that the sequences \( \{z_m\} \) and \( \{z'_m\} \) are uniformly bounded and equicontinuous, and \( \{Q_m\} \) weakly converges to some matrix \( Q(t) \). Using the fact that the set of symmetric \( n \times n \) matrices \( S \) satisfying \( A \leq S \leq B \) can be considered as a compact convex subset of \( \mathbb{R}^p, p = n(n+1)/2 \), it follows from Lemma 1A of (p. 157 of [5]) that \( Q(t) \) is a 2\( \pi \)-periodic symmetric matrix and \( A \leq Q(t) \leq B \). It is then shown that this leads to a contradiction of Theorem 1 of [1].

REFERENCES