EXISTENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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The results announced here concern the existence of a solution to the general initial value problem

\[(1) \quad x'(t) = f(t, x(t)), \quad x(0) = x_0,\]

in which \(x(t)\) lies in a Banach space \(X\) for \(t \in I = [0, a]\). Recent results for this problem have been announced in this Bulletin by S. N. Chow and J. D. Schuur [1] and by W. E. Fitzgibbon [2]. Related results were obtained earlier by F. Browder [3]. Here however, \(X\) is not assumed to be separable or reflexive, although as usual \(f\) will be continuous in \(x\) with respect to the weak topology on \(X\).

A pseudo-solution of (1) is an absolutely continuous function \(x: I \to X\) with pseudo-derivative (see Pettis [4]) satisfying (1). A strong solution of (1) is a strongly absolutely continuous function \(x: I \to X\) with strong derivative (\(\lim_{h \to 0} (x(t+h) - x(t))/h\) in norm) satisfying (1) a.e. on \(I\). For notions of absolute continuity, see Hille and Phillips [5, p. 76].

In what follows let \(B\) denote an open ball about some point \(x_0 \in X\), let \(I = [0, b]\) be a compact interval, and let \(f\) be a function from \(I \times B\) into \(X\).

**Theorem A.** Assume these hypotheses:

(a) For a.e. \(t \in I\), \(f(t, x)\) is continuous in the variable \(x\) with respect to the weak topology on \(B\) and \(X\).

(b) For each strongly absolutely continuous function \(y: I \to B\), \(f(t, y(t))\) is Pettis integrable on \(I\).

(c) For some null set \(N \subset I\), the weak closure of \(f((I-N) \times B)\) is weakly compact in \(X\).

Then (1) has a (possibly nonunique) pseudo-solution on a subinterval \(J = [0, a]\) of \(I\).

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The proof of Theorem A applies the Schauder-Tychonoff fixed point theorem to the transformation $T$ defined by

$$Ty(t) = x_0 + \int_0^t f(s, y(s)) \, ds \quad \text{(Pettis integral)}$$

on the intersection of certain convex subsets of the locally convex product space $X^J$ of all functions from $J$ into $X$.

Sufficient conditions for (b) to hold are that $X$ be weakly sequentially complete, that $f(t, y(t))$ be weakly measurable for each strongly absolutely continuous function $y: I \to B$, and that (c) hold. If we require that $f(t, y(t))$ be strongly measurable, we obtain the existence of a strong solution.

**Corollary B.** In Theorem A, replace condition (b) by the following condition.

(b*)

For every strongly absolutely continuous function $y: I \to B$, $f(t, y(t))$ is strongly measurable on $I$.

(This condition, together with (c), implies (b).)

Then every pseudo-solution of (1) is in fact a strong solution.

A simple sufficient condition for (b*) to hold is that for each point $x \in B$, $f(t, x)$ be strongly measurable with respect to $t$ on $I$.

Strong solutions of (1) will also be obtained in Theorem A if $X$ is uniformly convex, for pseudo-solutions of (1) under hypothesis (c) are in fact strongly absolutely continuous, hence strongly differentiable by Clarkson [6].

**Bibliography**


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