ON LOBATCHEWSKY MANIFOLDS

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Let $M$ be a complete, simply connected, $n$-dimensional Riemannian manifold with sectional curvature $K \leq 0$. Eberlein in [7] and [9] has given the cone topology and a nice compactification $\bar{M} = M \cup M(\infty)$ of $M$. The boundary $M(\infty)$ of $M$ is the set of asymptotic classes of geodesics in $M$. $\bar{M}$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$ and $M(\infty)$ is homeomorphic to $S^{n-1}$. Each isometry $\phi$ of $M$ extends to a homeomorphism of $\bar{M}$. Elements of the isometry group $I(M)$ can be classified according to their fixed points in $\bar{M}$. $\phi$ is called elliptic if $\phi$ has a fixed point in $M$. $\phi$ is called parabolic or axial if $\phi$ has exactly one fixed point or two fixed points in $M(\infty)$ respectively. If any two distinct points in the boundary $M(\infty)$ can be joined by a unique geodesic in $M$ (Axioms I and II), then $M$ is called a Lobatchewsky manifold for convenience. A complete, simply connected Riemannian manifold with sectional curvature $K \leq c < 0$ is a Lobatchewsky manifold.

In the sequel, we shall consider only Lobatchewsky manifolds $M$ and we shall assume that $I(M)$ acts effectively on $M$.

The main theorem is a description of complete homogeneous Riemannian manifolds with sectional curvature $K \leq c < 0$.

**Theorem 1.** Let $M$ be a complete homogeneous Riemannian manifold with sectional curvature $K \leq c < 0$. Either $I(M)$ has a common fixed point in $M(\infty)$ or $M$ is a noncompact symmetric space of rank one.

The tool of this paper is the concept of the limit set of a subgroup $G$ of $I(M)$. The limit set $L(G)$ is the intersection with $M(\infty)$ of the closure of any orbit of $G$ in $M$. The limit set is independent of the choice of the orbit. If $A$ is a closed subset of $M(\infty)$ which contains more than one point and $A$ is invariant under a subgroup $G$ of $I(M)$, then $A \cap L(G)$. The totally geodesic hull $\langle A \rangle$ of a subset $A$ in $M(\infty)$ is the intersection of all totally geodesic submanifolds in $M$ whose boundaries contain $A$.

Let $G$ be a subgroup of $I(M)$. One obtains classification of $L(G)$ in the following manner: (1) $L(G)$ is empty, (2) $L(G)$ contains one point,
(3) \(L(G)\) contains two points, (4) \(L(G)\) is an infinite, perfect and nowhere dense subset of \(M(\infty)\), (5) \(L(G)=M(\infty)\). Consequently, one can obtain classification of subgroups of \(I(M)\) according to their limit sets. A concrete classification of connected Lie subgroups of simple Lie groups of rank one has been accomplished in [5] and [6].

Here we present a unified version of the result in [5], [6] independent of Cartan's classification.

**Theorem 2.** Let \(M\) be a noncompact symmetric space of rank one. Let \(G\) be a connected Lie subgroup of \(I_0(M)\). Then one of the following holds:

1. \(G\) has a common fixed point in \(M\);
2. \(G\) has a common fixed point in \(M(\infty)\);
3. \(G\) modulo a normal subgroup (isomorphic to a subgroup of \(O(n-1)\)) is the 1-parameter group of axial elements\(^8\) along the geodesic joining two fixed points;
4. \(G\) modulo a normal subgroup (isomorphic to a subgroup of \(O(n-m)\), \(m=\dim(L(G))\)) is the connected isometry group \(I_0(S)\) of the totally geodesic submanifold \(S=\langle L(G) \rangle\) which is a noncompact symmetric space of rank one;
5. \(G=I_0(M)\).

A consequence of Theorem 2 is the following

**Theorem 3.** Let \(M\) be a noncompact symmetric space of rank one and \(G\) be a subgroup of \(I_0(M)\). If there is no point in \(M\) and no proper totally geodesic submanifold in \(M\) invariant under \(G\), then \(G\) is either discrete or dense in \(I_0(M)\).

The above fact is related to Borel's density theorem [4] and Selberg's irreducible lattices [19].

We outline the proof of Theorems 1 and 2 by stating two main lemmas.

**Lemma 1.** Let \(M\) be a simply connected complete Riemannian manifold with \(K\leq c<0\) such that \(I(M)\) acts effectively on \(M\). Suppose that \(G\) is a subgroup of \(I(M)\) and \(\langle L(G) \rangle = M\). If \(L(G)\) contains more than two points, then the centralizer \(Z(G, I(M))\) of \(G\) in \(I(M)\) is trivial. If, in addition, \(G\) does not have a common fixed point in \(M(\infty)\), then \(G\) is semisimple.

**Lemma 2.** Let \(M\) be a noncompact symmetric space of rank one and \(G\) be a Lie subgroup of \(I_0(M)\) such that \(M=\langle L(G) \rangle\). Suppose that \(L(G)\) contains more than two points and \(G\) does not have a common fixed point in \(M(\infty)\). Then either \(G\) is discrete or \(G=I_0(M)\).

\(^8\) The factored out normal subgroup of \(G\) contains elliptic elements which leave the geodesic pointwise fixed but may rotate other points in \(M\).
Finally we state a theorem on the density of axial fixed points for Lobatchewsky manifolds. This fact is indispensable to geodesic and horospherical $G$-partition flows on a Lobatchewsky manifold. One can easily obtain a straightforward generalization of [10]. Furthermore one gets a corollary which generalizes a theorem [8] of Eberlein.

**Theorem 4.** Let $M$ be a Lobatchewsky manifold and $G$ be a subgroup of $I(M)$. If $G$ contains axial elements and $G$ does not have a common fixed point in $M(\infty)$, then the fixed points of axial elements of $G$ are dense in $L(G) \times L(G)$.

**Corollary.** Let $M$ be a Lobatchewsky manifold and $G$ be a subgroup of $I(M)$. If $G$ does not have a common fixed point in $M(\infty)$ and $L(G)$ contains more than two points, then $G$ contains a free group with an infinite number of generators.

**References**


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