PARAMETRICES AND ESTIMATES FOR THE $\bar{\partial}_b$
COMPLEX ON STRONGLY PSEUDOCONVEX
BOUNDARIES

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0. Introduction. Here we briefly sketch the background of the problem
to be considered, and refer to Folland-Kohn [4] for definitions and proofs.

Let $X$ be the boundary of a strongly pseudoconvex region in a complex
manifold of complex dimension $n+1$, or more generally a real manifold
of dimension $2n+1$ with a strongly pseudoconvex partially complex
structure. We then have the tangential Cauchy-Riemann complex

$$0 \rightarrow \Lambda^0.0 \rightarrow \Lambda^0.1 \rightarrow \cdots \rightarrow \Lambda^0.n \rightarrow 0$$

where $\Lambda^0.j$ is the space of $j$-forms of purely antiholomorphic type. If we
impose a Riemannian metric on $X$, we can form the formal adjoint $\partial_b$ of $\bar{\partial}_b$ and thence the Laplacian $\Box_b = \bar{\partial}_b \partial_b + \partial_b \bar{\partial}_b$. $\Box_b$ is nonelliptic; however, according to a theorem of Kohn, for $1 \leq j \leq n-1$, $\Box_b$ satisfies the estimates

$$(i) \quad \| \phi \|_{s+1} \leq c_s(\| \Box_b \phi \|_s + \| \phi \|_0), \quad s = 0, 1, 2, \cdots ,$$

for all $\phi \in \Lambda^0.j$ with compact support. (Here $\| \|_s$ is the $L^2$ Sobolev norm of order $s$.) These estimates imply that $\Box_b$ is hypoelliptic; moreover, if $X$
is compact, the nullspace $\mathcal{N}$ of $\Box_b$ is finite-dimensional and there is an
operator $G$ on $\Lambda^0.j$ satisfying

$$(1) \quad \| G\phi \|_{s+1} \leq c_s \| \phi \|_s \quad (\phi \in \Lambda^0.j, s = 0, 1, 2, \cdots )$$

and

$$G \Box_b = \Box_b G = I - P$$

where $P$ is the orthogonal projection onto $\mathcal{N}$.

Kohn's method unfortunately gives no clue as to how to compute $G$.
Our purpose here is to construct $G$ (modulo smoothing operators) as an

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explicit integral operator and to derive sharp estimates for $\delta_b$ from this representation. Our method will be to construct an exact fundamental solution for $\Box_b$ on a particular manifold—which incidentally yields some interesting examples of hypoelliptic behavior—and then to transfer this solution to a general $X$.

1. **Analysis on the Heisenberg group.** Let $\mathcal{N} \subset \mathbb{C}^{n+1}$ be the real hypersurface

$$\mathcal{N} = \left\{ \zeta \in \mathbb{C}^{n+1} : \sum_{j=1}^{n} |\zeta_j|^2 = \Im \zeta_0 \right\}$$

$\mathcal{N}$ is the boundary of the generalized upper half-plane $\{ \zeta : \sum_{j=1}^{n} |\zeta_j|^2 < \Im \zeta_0 \}$, which is holomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. We take $(x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ as coordinates on $\mathcal{N}$ where $x_j = \Re \zeta_j$, $y_j = \Im \zeta_j$, $t = \Re \zeta_0$; we also write $z_j = x_j + iy_j$ and $z = (z_1, \ldots, z_n)$.

$\mathcal{N}$ is strongly pseudoconvex; moreover, $\mathcal{N}$ has a natural identification with a nilpotent Lie group (the Heisenberg group; cf. [7]). The group law is given by

$$(z, t)(z', t') = \left( z + z', t + t' + 2 \Im \sum_{j=1}^{n} z_j z_j' \right).$$

It is easy to verify that

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

form a basis for the Lie algebra of $\mathcal{N}$. Also, the forms $dz_1, \ldots, dz_n$ are a left-invariant basis for the antiholomorphic one-forms on $\mathcal{N}$.

$\delta_b$ is a left-invariant operator on $\mathcal{N}$, and it is not hard to compute it explicitly. If we set $Z_j = \frac{1}{2}(X_j - iY_j) = (\partial/\partial z_j) + i\bar{z}_j(\partial/\partial t)$, then

$$\delta_b \left( \sum_{J} \phi_J \, dz^J \right) = \sum_{k=1}^{n} (Z_k \phi_J) \, dz_k \wedge dz^J.$$

Here $J$ is a multi-index and $dz^J$ denotes a wedge product of $dz$'s.

We impose the left-invariant metric on $\mathcal{N}$ which makes $Z_1, \ldots, Z_n$, $Z_1, \ldots, Z_n$, $T$ orthonormal. Straightforward computation shows that the action of $\Box_b$ on $\Lambda^{0, j}$ is given by

$$\Box_b \left( \sum_J \phi_J \, dz^J \right) = - \sum_J (\mathcal{L}_{n-2} \phi_J) \, dz^J$$

where, for $\alpha \in \mathbb{C}$,

$$\mathcal{L}_\alpha = \frac{1}{2} \sum_{k=1}^{n} (Z_k Z_k + Z_k Z_k) - i\alpha T.$$
The study of $\Box_b$ is therefore reduced to the study of the left-invariant scalar operators $\mathcal{L}_\alpha$, $\alpha=n, n-2, \cdots, -n$.

We introduce the norm function $\rho(z,t) = (|z|^4 + t^2)^{1/4}$ on $N$, which arises naturally in the study of singular integrals on $N$ [6]. In [3] Folland showed that there is a constant $c_0 \neq 0$ such that $c_0^{-1} \rho^{-2n}$ is a fundamental solution for $\mathcal{L}_0$. From homogeneity and symmetry considerations it is then natural to search for a fundamental solution for $\mathcal{L}_\alpha$ of the form $\phi_\alpha(z,t) = \rho^{-2n}(z,t)f(t/\rho^2)$. The equation $\mathcal{L}_\alpha \phi_\alpha = \delta$ (where $\delta$ is the point mass at 0) leads to an ordinary differential equation for $f$ which can be solved explicitly, and the candidate for the fundamental solution turns out to be

$$\phi_\alpha(z,t) = (t + i|z|^2)^{-(n+\alpha)/2}(t - i|z|^2)^{-(n-\alpha)/2}.$$

**Theorem 1.**

$$\mathcal{L}_\alpha \phi_\alpha = c_\alpha \delta \quad \text{where} \quad c_\alpha = \frac{-i^{-2}2^{-2n}n^{n+1}}{\Gamma(\frac{1}{2}(n + \alpha))\Gamma(\frac{1}{2}(n - \alpha))}.$$

**Corollary.** $\mathcal{L}_\alpha$ is hypoelliptic if and only if $\pm \alpha \neq n, n+2, n+4, \cdots$.

For, if $\pm \alpha \neq n, n+2, n+4, \cdots$, then $c_\alpha \neq 0$ and $c_\alpha^{-1} \phi_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$ which is $C^\infty$ away from 0, whence $\mathcal{L}_\alpha$ is hypoelliptic. Otherwise, $c_\alpha = 0$, so that $\phi_\alpha$ is a nonsmooth solution of $\mathcal{L}_\alpha \phi_\alpha = 0$.

The family of operators $\mathcal{L}_\alpha$ bears some resemblance to an example of Grusin [5] which also involves hypoellipticity of an operator for "almost all" values of a parameter.

The occurrence of the "bad values" of $\alpha$ can be explained in terms of the representation theory of $N$. According to the Stone-von Neumann theorem, for each real $\lambda \neq 0$ there is a unique irreducible representation $\pi_\lambda$ of $N$ on $L^2(R^n)$ such that $\pi_\lambda(X_j) = -\partial/\partial \xi_j$, $\pi_\lambda(Y_j) = 4i\lambda \xi_j$, $\pi_\lambda(T) = i\lambda$ where $\xi_1, \cdots, \xi_n$ are coordinates on $R^n$, and $L^2(N)$ is a direct integral of these representations. (See [2].) Setting $\eta = 2|\lambda|^{1/2} \xi$, we have

$$\pi_\lambda(\mathcal{L}_\alpha) = |\lambda| \sum_{i=1}^n [(\partial^2/\partial \eta_i^2) - \eta_i^2] + \lambda \alpha.$$

Thus $\pi_\lambda(\mathcal{L}_\alpha)$ is invertible for (almost) all $\lambda$ if and only if $\pm \alpha$ is not an eigenvalue of the $n$-dimensional Hermite operator $\sum_{i=1}^n [\eta_i^2 - (\partial^2/\partial \eta_i^2)]$. But these eigenvalues are well known to be $n, n+2, n+4, \cdots$.

If $\alpha$ is not an exceptional value, the equation $\mathcal{L}_\alpha u = f$ is solved for reasonable $f$ by $u = f * (c_\alpha^{-1} \phi_\alpha)$, where $*$ denotes convolution on the group $N$. We can use this fact to derive sharp versions of the estimates (1) for $\mathcal{L}_\alpha$.

If $U \subset N$ is open, $1 \leq p \leq \infty$, $k \in R$, let $L^p_k(U)$ be the $L^p$ Sobolev space of order $k$ on $U$. For $k=0, 1, 2, \cdots$, we define $S^p_k(U)$ to be the space of all
$f \in L^p_{\nabla^2}(U)$ such that $D^\gamma f \in L^p(U)$ for all $|\gamma| \leq k$ where
\[
D = (X_1, \cdots, X_n, Y_1, \cdots, Y_n).
\]
$S^p_k$ has an obvious norm.

**Theorem 2.** Given $U \subset N$, $V \subset U$, $\pm x \neq n, n+2, n+4, \cdots$ and $f$ a function on $U$, let $u$ be any solution of $\mathcal{L}_u = f$ on $U$. If $f \in S^p_k(U)$ and $1 < p < \infty$ then $u \in S^p_{k+2}(V)$; also, if $f \in L^p(U)$, $q^{-1} = p^{-1} - (n+1)^{-1}$, and $1 < p < q < \infty$, then $u \in L^q(V)$.

The essential point of the proof is the fact that the distribution derivatives $D^\gamma \phi_x$ ($|\gamma| = 2$) and $T\phi_x$ are singular integral kernels à la Knapp-Stein [6] (plus, perhaps, multiples of $\delta$), and the corresponding convolutions are known to be bounded on $L^p$, $1 < p < \infty$ (cf. [1], [7]). The $L^p - L^q$ estimates were announced in Stein [8].

2. General strongly pseudoconvex manifolds. Let $X$ be a strongly pseudoconvex $(2n+1)$-manifold as in §0. We choose a nonvanishing real vector field $T$ which is complementary to the complex directions on $X$, so that $CTX = T_{1,0}X \oplus T_{0,1}X \oplus C \cdot T$. Replacing $T$ by $-T$ if necessary, the Levi form $\langle , \rangle$ on $T_{1,0}X$ given for $Z_1, Z_2 \in C^\infty(T_{1,0}X)$ by
\[
[Z_1, Z_2] = -2i\langle Z_1, Z_2 \rangle T \text{ modulo } C^\infty(T_{1,0}X \oplus T_{0,1}X)
\]
is positive definite. We extend $\langle , \rangle$ to a Hermitian metric on $X$ by requiring $T_{1,0}X \perp T_{0,1}X \perp T$ and $\langle T, T \rangle = 1$, and consider the Laplacian $\Box_b$ associated to this metric. We work locally and fix once and for all an orthonormal frame $Z_1, \cdots, Z_n$ for $T_{1,0}X$. Further we denote the dual frame for $T^*_{1,0}X$ by $\omega_1, \cdots, \omega_n$.

In this setup $X$ looks locally like the Heisenberg group modulo small error terms, in the sense provided by the following two lemmas.

**Lemma 1.** If $\phi = \sum_j \phi_j \omega^j \in \Lambda^{0,j}$, then
\[
\Box_b \phi = \sum_j \left( -\frac{1}{2} \sum (Z_k \bar{Z}_k + Z_k \bar{Z}_k) + (n - 2j)iT[\phi_j]\omega^j \right)
\]
modulo terms of order one and zero not involving differentiation in the $T$ direction.

**Lemma 2.** For each $\xi \in X$ there exist local coordinates $x_1^\xi, \cdots, x_n^\xi, y_1^\xi, \cdots, y_n^\xi, t^\xi$ on a neighborhood $U_\xi$ of $\xi$, which are centered at $\xi$ and depend smoothly on $\xi$, such that with $z^\xi_k = x_k^\xi + iy_k^\xi$, on $U_\xi$ the vector fields $Z_k$ and $T$ take the form
\[
Z_k = \frac{\partial}{\partial z_k^\xi} + iz_k^\xi \frac{\partial}{\partial t^\xi} + \sum \left( a_{km} \frac{\partial}{\partial z_m^\xi} + b_{km} \frac{\partial}{\partial z_m^\xi} \right) + c_k \frac{\partial}{\partial t^\xi},
\]
\[
T = \frac{\partial}{\partial t^\xi} + \sum \left( a_m \frac{\partial}{\partial z_m^\xi} + \beta_m \frac{\partial}{\partial z_m^\xi} \right) + \gamma \frac{\partial}{\partial t^\xi}
\]
where \( a_{km}, b_{km}, \alpha_m, \beta_m, \) and \( \gamma \) vanish to first order at \( \xi \), and \( c_k \) vanishes to first order in \( t^k \) and to second order in \( z_m^e \) and \( \bar{z}_m^e \), \( m=1, \ldots, n \).

These coordinates are constructed using exponentials of linear combinations of \( Z_k, Z_k^*, T \). In case \( X \) is realized as a hypersurface in a complex manifold \( M \), we can also construct them by restricting certain distinguished holomorphic coordinates on \( M \) to \( X \).

We can now construct a parametrix for \( \Box_b \) on \( \Lambda^{0,j} \), \( 1 \leq j \leq n-1 \). By applying a partition of unity it suffices to consider forms supported in a fixed compact set \( V \). Let \( \Omega = \{ (\eta, \xi) \in X \times X : \eta \in U_\xi \} \), and choose \( \psi \in C_0^\infty(\Omega) \) which = 1 on a neighborhood of the diagonal in \( V \times V \). Define the double form \( K_j \in \Lambda^{0,j} \otimes \Lambda^{2n+1-j} \) by

\[
K_j(\eta, \xi) = -c_{m-2j}^{-1} \psi(\eta, \xi)(t^k(\eta) + i |z^k(\eta)|^2)^{j-n} \times (t^k(\eta) - i |z^k(\eta)|^{j-k} \sum J \bar{\partial}^J(\eta) \otimes (\bar{\partial}^J)(\xi).
\]

Define the operator \( K \) on \( \{ \phi \in \Lambda^{0,j} : \text{supp } \phi \subset V \} \) by

\[
K \phi(\eta) = \int_{\xi} K_j(\eta, \xi) \wedge \phi(\xi),
\]
and set \( S = I - \Box_b K \). With the Sobolev spaces \( S^p_k = S^p_k(V) \) defined as in §1, we then have

**Theorem 3.** \( K \) is bounded from \( S^p_k \) to \( S^p_{k+2} \) (\( 1 < p < \infty \)) and from \( L^p \) to \( L^q \) (\( q^{-1} = p^{-1} - (n+1)^{-1}, 1 < p < q < \infty \)). \( S \) is bounded from \( S^p_k \) to \( S^p_{k+1} \) (\( 1 < p < \infty \)) and from \( L^p \) to \( L^q \) (\( q^{-1} = p^{-1} - \frac{1}{2}(n+1)^{-1}, 1 < p < q < \infty \)).

**Corollary.** \( I - \Box_b K(\sum_{m=0}^{m-1} S^m) = S^m \) is bounded from \( S^p_k \) to \( S^p_{k+m} \).

Thus we have a right inverse to \( \Box_b \) modulo smoothing operators of arbitrarily high order. The corresponding left inverse is obtained by using the adjoint operator \( K^* \); the analogues of Theorem 3 and its corollary hold here also. (The main point is to observe that the coordinates of Lemma 2 are essentially symmetric in \( \xi \) and \( \eta \).)

It is also possible to obtain estimates for \( K \) and \( S \) in terms of the non-isotropic Lipschitz norms introduced in Stein [8].

Details and proofs will appear in a later publication.

**References**


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