INTERPOLATION OF OPERATORS FOR \( \Lambda \) SPACES

BY ROBERT SHARPLEY

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Lorentz and Shimogaki [2] have characterized those pairs of Lorentz \( \Lambda \) spaces which satisfy the interpolation property with respect to two other pairs of \( \Lambda \) spaces. Their proof is long and technical and does not easily admit to generalization. In this paper we present a short proof of this result whose spirit may be traced to Lemma 4.3 of [4] or perhaps more accurately to the theorem of Marcinkiewicz [5, p. 112]. The proof involves only elementary properties of these spaces and does allow for generalization to interpolation for \( n \) pairs and for \( M \) spaces, but these topics will be reported on elsewhere.

The Banach space \( \Lambda_\phi \) [1, p. 65] is the space of all Lebesgue measurable functions \( f \) on the interval \( (0, l) \) for which the norm

\[
\| f \|_\phi = \int_0^l f^*(s) \phi(s) \, ds
\]

is finite, where \( \phi \) is an integrable, positive, decreasing function on \( (0, l) \) and \( f^* \) (the decreasing rearrangement of \( |f| \)) is the almost-everywhere unique, positive, decreasing function which is equimeasurable with \( |f| \).

A pair of spaces \( (\Lambda_\phi, \Lambda_\psi) \) is called an interpolation pair for the two pairs \( (\Lambda_{\phi_1}, \Lambda_{\psi_1}) \) and \( (\Lambda_{\phi_2}, \Lambda_{\psi_2}) \) if each linear operator which is bounded from \( \Lambda_{\phi_i} \) to \( \Lambda_{\psi_i} \) (both \( i=1, 2 \)) has a unique extension to a bounded operator from \( \Lambda_\phi \) to \( \Lambda_\psi \).

**THEOREM (LORENTZ-SHIMOGAKI).** A necessary and sufficient condition that \( (\Lambda_\phi, \Lambda_\psi) \) be an interpolation pair for \( (\Lambda_{\phi_1}, \Lambda_{\psi_1}) \) and \( (\Lambda_{\phi_2}, \Lambda_{\psi_2}) \) is that there exist a constant \( A \) independent of \( s \) and \( t \) so that

\[
(\ast) \quad \frac{\Psi(t)/\Phi(s)}{\Phi(s)} \leq A \max_{i=1,2} (\psi_i(t)/\Phi_i(s))
\]

holds, where \( \Phi(s) = \int_0^s \phi(r) \, dr \), \( \cdots \), \( \psi_2(t) = \int_0^t \Psi_2(r) \, dr \).

**PROOF.** We only sketch the proof of the necessity since it is standard.


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Suppose there are numbers $s_n$ and $t_n$ in $(0, l)$ such that $\Psi(t_n)/\Phi(s_n) > n^3 \max_{i=1,2}(\Psi_i(t_n)/\Phi_i(s_n))$. Define the positive operator

$$T_n f(t) = \left( C_n \int_0^{s_n} f(s)/s_n \right) \chi(0,t_n)(t),$$

where $C_n = \min_{i=1,2}(\Phi_i(s_n)/\Psi_i(t_n))$.

For each $f$ in $A^\phi$, $T_n f$ belongs to $A^\psi_i$ and $T_n$ has operator norm less than or equal to 1, but as an operator from $A^\phi$ to $A^\psi$, $T_n$ has operator norm larger than $n^3$. Hence the operator $T = \sum_{i=1}^\infty T_n/n^2$ is a bounded operator from $A^\phi_i$ to $A^\psi_i$ $(i=1, 2)$, but $T$ is not a bounded operator from $A^\phi$ to $A^\psi$.

To show that condition (*) is sufficient, we prove that

$$\|T f\|_\psi \leq 2M \|f\|_\phi$$

where $M$ is the maximum of the operator norms of $T$ acting from $A^\phi_i$ to $A^\psi_i$ $(i=1, 2)$. We can assume that $f$ is an arbitrary simple function with finite support since these functions are dense in $A^\phi$. We can also require $f$ to be positive since $\|f\|_\phi = \|f\|_\psi$. Each function of this type can be written as $f = \sum_i a_i \chi E_i$ where the $a_i$'s are positive and $E_n \subseteq \cdots \subseteq E_1$. Hence $f^* = \sum_i a_i \chi (0,a_i)$ where $a = mE_i$. But then

$$\|T f\|_\psi \leq 2AM \Phi(mE), \quad \forall \text{ measurable } E \subset (0, l)$$

is equivalent to relation (1), since

$$\|T f\|_\psi \leq \sum_{i=1}^n \alpha_i \|T \chi E_i\|_\psi \leq 2AM \sum_{i=1}^n \alpha_i \Phi(a) = 2AM \|f\|_\phi .$$

Hence, if we let $g = (T \chi E)^*$, the proof is reduced to the following

**Lemma.** Suppose condition (*) holds and $g$ is a positive decreasing function that satisfies

$$\|g\|_\psi \leq M \Phi_i(a) \quad (i = 1, 2),$$

then

$$\|g\|_\psi \leq 2AM \Phi(a).$$

**Proof.** First, assume $g$ is a step function with finite support, i.e., $g = \sum_{i=1}^n \beta_i \chi (0,t_i)$. Set $J = \{j|\max_{i=1,2}(\Psi_i(t_j)/\Phi_i(a)) = \Psi_1(t_j)/\Phi_1(a)\}$ and then let $g_1 = \sum_{j \in J} \beta_i \chi (0,t_i)$ and $g_2 = g - g_1$. Notice that both functions are positive, decreasing, step functions and

$$\|g_i\| \leq \|g\|$$
in any $\Lambda$ space. Now using condition ($\ast$), relations (5) and (3), we have

$$\|g_1\|_{\psi_i}/\Phi(a) = \sum_{i \in I} \beta_i \Psi(t_i)/\Phi_i(a)$$

$$\leq A \sum_{i \in I} \beta_i \max_{i=1,2}(\Psi_i(t_i)/\Phi_i(a))$$

$$= A \sum_{i \in I} \beta_i \Psi_i(t_i)/\Phi_i(a) = A \|g_1\|_{\psi_i}/\Phi_i(a)$$

$$\leq A \|g\|_{\psi_i}/\Phi_i(a) \leq A M.$$  

Similarly

$$\|g_2\|_{\psi_i}/\Phi(a) \leq A \|g_2\|_{\psi_i}/\Phi_2(a) \leq A M.$$  

Hence, we obtain relation (4) for positive, decreasing, step functions.

Now suppose $g$ is an arbitrary positive decreasing function and let $\{g_n\}$ be a monotone increasing sequence of positive decreasing step functions converging pointwise to $g$. By (3) and (5)

$$\|g_n\|_{\psi_i} \leq M\Phi_i(a) \quad (i = 1, 2)$$

so

$$\|g_n\|_{\psi} \leq 2AM\Phi(a).$$

Applying the monotone convergence theorem to $\{g_n\}$, we obtain relation (4).

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REFERENCES


DEPARTMENT OF MATHEMATICS, OAKLAND UNIVERSITY, ROCHESTER, MICHIGAN 48063