PROPERTIES OF THREE ALGEBRAS RELATED TO $L^p$-MULTIPLIERS

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Communicated by J. L. Kelley, July 2, 1973

1. Introduction. In this paper we shall announce several properties of certain algebras which arise in the study of $L^p$-multipliers; detailed proofs will be given elsewhere. Let $G$ be a locally compact abelian group and let $\Gamma$ denote its dual group. Let $L^p(\Gamma)$ denote the space of $p$-integrable functions on $\Gamma$ with respect to Haar measure, and let $q$ denote the index which is conjugate to $p$. Let

$$A_p(\Gamma) = [L_p(\Gamma) \hat{\otimes} L_q(\Gamma)]/K$$

where $K$ is the kernel of the convolution operator $c: L_p \hat{\otimes} L_q(\Gamma) \to C_0(\Gamma)$ by $c(f \otimes g)(\gamma) = (f \ast g)(\gamma)$. $A_p(\Gamma)$ is the $p$-Fourier algebra which was introduced by Figa-Talamanca in [6] where it was shown that $A_p(\Gamma)^*$ is isometrically isomorphic to $M_p(\Gamma)$, the bounded, translation invariant, linear operators on $L^p(\Gamma)$. Herz [11] showed that $A_p(\Gamma)$ is a Banach algebra under pointwise multiplication; it is known that $A_2(\Gamma) = A(\Gamma) = L_1(G)^\wedge$ and that the inclusions $A_2(\Gamma) \subseteq A_p(\Gamma) \subseteq A_1(\Gamma) = C_0(\Gamma)$ are norm decreasing if $1 < p < 2$; see [5], [6], [11] for the basic properties of $A_p(\Gamma)$. Let $B_p(\Gamma)$ denote the algebra of continuous functions $f$ on $\Gamma$ such that $f(\gamma)h(\gamma) \in A_p(\Gamma)$ whenever $h \in A_p(\Gamma)$. The multiplier algebra $B_p(\Gamma)$ is a Banach algebra in the operator norm. We have studied $B_p(\Gamma)$ in [8], [9].

Fix $p$ in $1 < p < 2$.

Regard $L_1(\Gamma)$ as an algebra of convolution operators on $L^p(\Gamma)$ and let $m_p(\Gamma)$ denote the closure of $L_1(\Gamma)$ in $M_p(\Gamma)$. The first result of this paper says that $B_p(\Gamma)$ is isometrically isomorphic to the dual space $m_p(\Gamma)^*$. In the second result, we use certain properties of $B_p(\Gamma)$ to give a theorem of Eberlein type for $M_p(\Gamma)$. In the final section of the paper, we use


Key words and phrases. $p$-Fourier algebra, multipliers, dual space representation, Eberlein's theorem.

1 Research supported in part by the National Science Foundation grant GP-24574.

* This paper is being published posthumously. Professor Michael J. Fisher died August 27, 1973. However, all correspondence concerning this paper should be addressed to the author at the address below.

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$m_p(\Gamma)$ to represent $M_p(\Gamma)$ as the multiplier algebra of a certain subalgebra of $M_p(\Gamma)$. For the case when $\Gamma$ is a compact group, Theorem 1 was proved in [1].

Our work has been influenced by Theorems 1.1 and 1.2 of [12]. There McKilligan studied the multiplier algebra $B$ of a commutative, semisimple, Banach algebra $A$ which contains an approximate identity of norm one. $A^{**}$ is equipped with the Arens product $(\cdot)$ and $B$ is isometrically embedded in $(A^{**}, (\cdot))$ by the mapping $T \rightarrow T^{**}(j)$ where $j$ is the right identity in $A^{**}$; see [2] for the basic properties of the Arens product. Thus if $T \in B$ and if $\{e_a\}$ is the approximate identity in $A$, then

$$T^{**}(j)(F) = \lim_{a} F(T(e_a))$$

for every functional $F \in A^*.$

We shall not distinguish between $M_p(\Gamma)$ and $A_p(\Gamma)^*$. If $H \in L_1(\Gamma)$, let $*H$ denote the corresponding convolution operator on $L_p(\Gamma)$. If $\psi \in m_p(\Gamma)^*$, let $\|\psi\|_{\psi}$ denote the norm of $\psi$. If $h \in A_p(\Gamma)$, $|h|_p$ denotes its norm; if $f \in B_p(\Gamma)$, $\|f\|_p$ is the operator norm; and if $T \in M_p(\Gamma)$, $\|T\|_p$ is the operator (or functional) norm of $T$. An approximate identity $\{E_a\}$ in $L_1(G)$ which consists of normalized characteristic functions of compact, symmetric neighborhoods of the identity is referred to as a standard approximate identity. The corresponding net $\{E_a\}$ in $A_2(\Gamma) = A(\Gamma)$ is also referred to as a standard approximate identity.

After we had submitted this paper for publication, Professor Carl Herz told us that he had given a different proof of Theorem 1 in 1970 for amenable groups. N. Lohoué, C.R. Acad. Sci. Paris 272 (1971) and 273 (1971), refers to Herz’s result as presented at Orsay in June, 1970.

2. Dual space representation.

**Theorem 1.** $B_p(\Gamma)$ is isometrically isomorphic to $m_p(\Gamma)^*$ by the map $\varphi \rightarrow \tilde{\varphi}$ when $\tilde{\varphi}(h) = \int_{\Gamma} \tilde{\varphi}(\gamma)H(\gamma) \, d\gamma$ for all $H \in L_1(\Gamma)$.

Use Theorems 1 of [6] and [7] to show that $h \rightarrow \tilde{h}$ gives an isometric embedding of $A_p(\Gamma)$ into $m_p(\Gamma)^*$. Use a standard approximate identity to extend this embedding to $B_p(\Gamma)$. Conversely, let $\tilde{\varphi} \in m_p(\Gamma)^*$; then there is a bounded measurable function $\varphi_0(\gamma)$ such that

$$\tilde{\varphi}(h) = \int_{\Gamma} \varphi_0(\gamma)H(\gamma) \, d\gamma.$$

Define

$$\psi_{ab}(\gamma) = (\varphi_0 E_a) \ast \varphi_b(\gamma)$$

when $\{E_a\}$ and $\{\varphi_b\}$ are standard approximate identities in $L_1(G)$ and $L_1(\Gamma)$ respectively. Then $|\psi_{ab}|_p \leq \|\psi\|_{\psi}$ and $\{\psi_{ab}\}$ converges to $\varphi_0$ in the weak*
topology of \( L^\infty(\Gamma) \). Let \( \mathcal{B}_p \) denote the algebra of bounded measurable functions \( \psi \) on \( \Gamma \) for which \( M(\psi)(x, y) = \psi(xy^{-1}) \) is a multiplier on \( L_p \otimes L_q(\Gamma) \). By following Eymard [5], one shows that \( \mathcal{B}_p(\Gamma) = B_p(\Gamma) \).

Let \( E_q = L_p \otimes \lambda L_q(\Gamma) \), the completion of \( L_p \otimes L_q(\Gamma) \) with respect to the least cross norm. By a theorem of Grothendieck [10, p. 122], \( E_q^* = L_p \otimes L_q(\Gamma) \). Using this fact one shows that \( M(\psi_{ab}) \) converges to \( M(\psi_0) \) in the weak* topology of \( L_p \otimes L_q(\Gamma) \) and that \( \psi_0 \in \mathcal{B}_p(\Gamma) \).

By letting \( m_p(\Gamma)^* = M^\ast(\Gamma)^* / m_p(\Gamma)^\perp \) have the quotient Arens product, one sees that \( \varphi \mapsto \tilde{\varphi} \) is an algebra isomorphism as well.

3. Eberlein’s theorem. Use McKilligan’s representation for multipliers to regard a function \( f \in B_p(\Gamma) \) as a functional \( \tilde{f} \in M_p(\Gamma)^* \).

**Theorem 2.** Let \( M_p(\Gamma)_c \) denote the \( L_p \)-multipliers with continuous Fourier transforms. An operator \( T \in M_p(\Gamma)_c \) is in \( M_p(\Gamma)_c \) if and only if there is a constant \( M \geq 0 \) such that for every finite set \( \{a_k\} \) of complex numbers and every equinumerous subset \( \{g_k\} \subset G \), the Fourier transform \( \hat{T} \) of \( T \) satisfies

\[
\sum_{k=1}^n |a_k \hat{T}(g_k)| \leq M \left\| \sum_{k=1}^n a_k \hat{g}_k \right\|_p.
\]

When \( T \in M_p(\Gamma)_c \), \( \|T\|_p \) is the least constant \( M \) for which the inequality holds.

If \( T \in M_p(\Gamma)_c \), it follows from McKilligan’s representation that \( \hat{g}(T) = \hat{T}(g) \) for \( g \in G \), so that the inequality holds for some \( M \leq \|T\|_p \). By Saeki’s Theorem 4.3 of [14], \( \|T\|_p \) is the least constant \( M \) for which the inequality holds. If \( T \in M_2(\Gamma)_c \) satisfies the inequality, so does \( T_{ab} = *(\hat{f}_b T(E_a)) \) when \( \{f_b\} \subset L_1(G) \) and \( \{E_a\} \subset L_1(\Gamma) \) are standard approximate identities. Since \( \|T_{ab}\|_p \leq M \), the net \( \{T_{ab}\} \) has a weak* convergent subnet \( \{T_b\} \in M_p(\Gamma) \). Since \( A_2(\Gamma) = A(\Gamma) \) is dense in \( A_p(\Gamma) \), it follows that \( T = \lim T_b \in M_p(\Gamma) \).

From [14], a function \( F \in L^\infty(G) \) is said to be regulated if there is an approximate identity \( \{E_a\} \) of norm one in \( L_1(G) \) such that \( \{F * E_a\} \) converges pointwise and in the weak* topology of \( L^\infty(G) \) to \( F \). Theorem 2 can be extended so as to apply to operators with regulated Fourier transforms.

The separation theorem [3, p. 417] for compact convex sets and Theorem 2 now imply

**Theorem 3.** If \( f \in B_p(\Gamma) \), there is a net \( \{f_b\} \) in the span of \( G \) in \( B_p(\Gamma) \) such that \( \|f_b\|_p \leq \|f\|_p \) and such that \( \{f_b\} \) converges to \( f \) in the weak* topology of \( B_p(\Gamma) \).
4. **$M_p$ as a multiplier algebra.** Use multiplication of operators to regard $M_p(\Gamma)$ as an algebra over the ring $m_p(\Gamma)$. In particular, $M_p(\Gamma)$ is an $m_p(\Gamma)$-module. It follows from the general form of Cohen's factorization theorem [13, p. 453] that the $m_p$-essential submodule of $M_p(\Gamma)$ is

$$M_p m_p(\Gamma) = \{ K \in M_p(\Gamma) \mid K = UT, U \in M_p(\Gamma), T \in m_p(\Gamma) \}.$$ 

$M_p m_p(\Gamma)$ is a Banach algebra in the operator norm and a standard approximate identity in $L_1(\Gamma)$ is an approximate identity of norm one in $M_p m_p(\Gamma)$.

**THEOREM 4.** $M_p(\Gamma)$ is the algebra of multiplier operators on $M_p m_p(\Gamma)$.

A weak* compactness argument is used.

$M_p m_p(\Gamma)$ plays the role in $M_p(\Gamma)$ that $L_1(\Gamma)$ plays in $M(\Gamma)$.

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