FROBENIUS RECIPROCITY OF DIFFERENTIABLE REPRESENTATIONS

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Communicated by François Treves, July 10, 1973

ABSTRACT. In this note we give the construction of the adjoint and the coadjoint of the restriction functor in the category of differentiable $G$-modules, where $G$ is a Lie group.

1. Introduction. Let $G$ be a Lie group, countable at infinity. A continuous representation $\lambda$ of $G$ in a complete locally convex space $E$ is differentiable if for each $a \in E$ the map $\hat{a} : x \to \lambda(x)a$ of $G$ into $E$ is $C^\infty$, and if the injection $a \to \hat{a}$ of $E$ into $C^\infty(G,E)$ is a topological homeomorphism [8]. We then say that $E$ is a differentiable $G$-module.

There is a natural way of associating a differentiable representation to any continuous, in particular unitary, representation of $G$. In fact, let $\rho$ be a continuous representation of $G$ on a complete locally convex space $F$. Let $F_\infty = \{a \in F : \hat{a} \in C^\infty(G,F)\}$. Then $F_\infty$ is a dense $\rho$-invariant linear subspace of $F$. The injection $a \to \hat{a}$ sends $F_\infty$ onto a closed subspace of $C^\infty(G,F)$. When $F_\infty$ is equipped with the relative topology of $C^\infty(G,F)$ it becomes a complete locally convex space, and the corresponding subrepresentation $\lambda_\infty$ of $\lambda$ on $F_\infty$ is differentiable. If $\lambda$ is topologically irreducible then $\lambda_\infty$ is topologically irreducible and conversely. For details and other basic facts concerning differentiable representations see [8].

The purpose of the present note is to show that the Frobenius reciprocity theorem is valid in the category of differentiable $G$-modules. The history of the Frobenius reciprocity theorem is long and interesting. For some recent developments the reader is referred to the work of Bruhat [1], Moore [4], Rieffel [5] and Rigelhof [6]. In particular Rigelhof succeeded in constructing an adjoint and a coadjoint for the restriction functor in the category of continuous (locally convex) $G$-modules.

2. Construction of the adjoint and the coadjoint. Let $K$ be a closed subgroup of $G$, and let $F$ be a differentiable $G$-module. The restriction $F \to F_K$ is a functor from the category of differentiable $G$-modules to the category of differentiable $K$-modules.

Let $E$ be a differentiable $K$-module and let $\pi$ be the corresponding representation.

(1) Coadjoint functor. Let $E' = E'$ denote the space of distributions
with compact support on $G$, equipped with the strong topology as the dual of $C^\infty(G)$.

Let $E^G$ denote the space of all continuous $K$-linear maps of $\mathcal{E}'$ into $E$, i.e., $E^G = \text{Hom}_K(\mathcal{E}', E)$ and $m \in E^G$ iff

\begin{equation}
(2.1) \quad m(kS) = km(S); \quad S \in \mathcal{E}', \; k \in K.
\end{equation}

Here $(kS)(f) = S(fk)$, where $(fk)(x) = f(kx)$, $f \in C^\infty(G)$, $x \in G$. $E^G$ is given the topology of uniform convergence on compact sets and is a complete locally convex space. We define the induced representation $\pi^G$ of $G$ on $E^G$ by

\begin{equation}
(2.2) \quad [\pi^G(x)m](S) = m(Sx).
\end{equation}

**Proposition 1.** $\pi^G$ is a differentiable representation of $G$ on $E^G$.

(2) **Adjoint functor.** A bilinear map $\omega: \mathcal{E}' \times E \to H$, $H$ a locally convex space is $K$-balanced if $\omega(Sk, a) = \omega(S, ka)$ for all $S \in \mathcal{E}'$, $k \in K$, $a \in E$. (We let $K$ act to the right on $\mathcal{E}'$ here and write $ka$ for $\pi(k)a$.) Let $B_K(\mathcal{E}', E)$ denote the space of all $K$-balanced bilinear maps of $\mathcal{E}' \times E$ into $C$. Let $\chi: \mathcal{E}' \times E \to B_K(\mathcal{E}', E)^*$ be the canonical map:

\[ \chi(S, a)b = b(S, a); \quad b \in B_K(\mathcal{E}', E). \]

$\chi$ is $K$-balanced and bilinear. We let $\mathcal{E}' \otimes_K E$ denote the linear span of the range of $\chi$. Typical elements of $\mathcal{E}' \otimes_K E$ will be written $\sum_{i=1}^{n} S_i \otimes a_i$. We give $\mathcal{E}' \otimes_K E$ the inductive tensor product topology with respect to the family of bounded subsets of $\mathcal{E}'$ and $E$ [2]. Let $^G\mathcal{E}$ be the completion of $\mathcal{E}' \otimes_K E$ with respect to this topology. We define the representation $^G\pi$ of $G$ on $^G\mathcal{E}$ by

\begin{equation}
(2.3) \quad ^G\pi(x)(S \otimes a) = xS \otimes a.
\end{equation}

This construction is similar to the one given in [5] and [6].

**Proposition 2.** $^G\pi$ is a differentiable representation of $G$ on $^G\mathcal{E}$.

3. **Main result.** Preserve the notation and assumptions above.

**Theorem 1.** $E \rightarrow ^G\mathcal{E}$ is the adjoint and $E \rightarrow E^G$ is the coadjoint functor of the restriction functor $F \rightarrow F_K$, i.e., there are natural isomorphisms (in the sense of category theory):

\begin{align}
(3.1) \quad & \text{Hom}_G(^G\mathcal{E}, F) \cong \text{Hom}_K(E, F_K); \\
(3.2) \quad & \text{Hom}_G(F, E^G) \cong \text{Hom}_K(F_K, E).
\end{align}

Moreover, the adjoint and coadjoint are unique to within equivalence of differentiable $G$-modules.

**Remark.** The construction of the isomorphism in (3.1) rests upon the preliminary result that $\omega: (S, a) \rightarrow \lambda(S)a$ is a hypocontinuous bilinear map of $\mathcal{E}' \times F$ into $F$. Here $\lambda(S)$ denotes the distribution form of the
representation \( \lambda \), \( \omega \) is also \( K \)-balanced, and therefore any \( A \in \text{Hom}_K(E, F) \) determines a continuous linear map \( A' : \mathcal{E}' \otimes_K E \to F \) such that
\[
A' \sum S_i \otimes a_i = \sum \lambda(S_i) A a_i = \sum \omega(S_i, A a_i).
\]
The map \( A \mapsto A' \) defines (3.1).

For (3.2) let \( A \in \text{Hom}_K(F, E) \), and let \( a \in F \). Define \( (A' a)(S) = A \lambda(S)a \). Then \( A' a \) belongs to \( \text{Hom}_K(\mathcal{E}', E) = E^G \) and \( A' : a \to A' a \) belongs to \( \text{Hom}_G(F, E^G) \). The map \( A \to A' \) defines (3.2).

Finally, both (3.1) and (3.2) are topological isomorphisms (with respect to standard topologies).

4. Realizations. In this section we give alternative descriptions of the \( G \)-modules \( E^G \) and \( {}^G E \). As may be expected \( E^G \) may be realized as a space of \( E \)-valued \( C^\infty \)-functions. Let \( C_K^\infty(G, E) \) denote the space of \( C^\infty \)-functions \( f : G \to E \) satisfying
\[
f(kx) = \pi(k)f(x); \quad k \in K, \; x \in G.
\]
We give \( C_K^\infty(G, E) \) the relative topology from \( C^\infty(G, E) \), and let \( G \) act as the right regular representation on \( C_K^\infty(G, E) \). This makes \( C_K^\infty(G, E) \) into a differentiable \( G \)-module, and we have

**Proposition 3.** The differentiable \( G \)-modules \( E^G \) and \( C_K^\infty(G, E) \) are equivalent.

The proof is more or less straightforward, based on the isomorphisms \( \text{Hom}(\mathcal{E}', E) \cong C^\infty(G)^\mathcal{E} \cong E \cong C^\infty(G, E) \) (\( C^\infty(G) \) is a reflexive nuclear space; see [7].)

It does not appear possible to realize \( {}^G E \) as a space of functions. There is, however, another representation of \( {}^G E \), in a particular case, which throws some light on the connection between \( {}^G E \) and \( E^G \).

Let \( C_\infty^\infty(G) \) denote the space of complex-valued \( C^\infty \)-functions on \( G \) with compact support, equipped with the usual topology [7], [8]. Let \( \text{Hom}^0_K(C_\infty^\infty(G), E) \) be the space of all continuous linear maps \( m : C_\infty^\infty(G) \to E \) which satisfy
\[
supp m \subseteq CK \quad \text{for some compact subset} \; C \; \text{of} \; G;
\]
(4.2) \[
m(k\varphi) = \delta(k)^{-1} km(\varphi) \quad \text{where} \; k \in K, \; \varphi \in C_\infty^\infty(G), \; \text{and}
\]
(4.3) \[
\delta \; \text{is the modular function of} \; K.
\]

We topologize \( \text{Hom}^0_K(C_\infty^\infty(G), E) \) as follows. For each compact set \( C \), let \( \text{Hom}^C_K(C_\infty^\infty(G), E) \) be the subspace of those \( m \)'s that have their support in \( CK \). We give this space the relative topology as a subspace of \( \text{Hom}(C_\infty^\infty(G), E) \). \( \text{Hom}^0_K(C_\infty^\infty(G), E) \) is then given the inductive limit topology from the family of spaces \( \text{Hom}^C_K(C_\infty^\infty(G), E) \) as \( C \) runs through the collection of compact subsets of \( G \). We make \( \text{Hom}^0_K(C_\infty^\infty(G), E) \) into
a differentiable $G$-module by the action
\begin{equation}
(xm)\varphi = m(\varphi x); \quad \varphi \in C^\infty_0(G), \ x \in G.
\end{equation}

**Proposition 4.** If $E$ is the dual of a reflexive Fréchet space then the differentiable $G$-modules $\mathcal{E}$ and $\text{Hom}_{k}^{0}(C^\infty_0(G), E)$ are equivalent.

The proof of this result rests upon another proposition, stated below. Let $\text{Hom}^{0}(C^\infty_0(G), E)$ be the space of all continuous linear maps $m: C^\infty_0(G) \to E$ with compact support. We equip this space with the natural inductive topology. Let $K$ act to the right on the space by
\begin{equation}
(mk)\varphi = \delta(k)k^{-1}m(k\varphi).
\end{equation}
Then define
\begin{equation}
m^\# \varphi = \int_{K} (mk)\varphi \, dk; \quad k \in K, \ \varphi \in C^\infty_0(G).
\end{equation}

**Proposition 5.** The map $m \mapsto m^\#$ is linear, continuous, and open of $\text{Hom}^{0}(C^\infty_0(G), E)$ onto $\text{Hom}_{K}^{0}(C^\infty_0(G), E)$.

**Remark.** This result is true when $E$ is a complete locally convex space.

5. **Concluding remarks.** Bruhat [1] has given another definition of differentiably induced representations, and has given a version of the Frobenius reciprocity theorem in terms of intertwining forms. Full proofs of the results above, a discussion of the relationship to Bruhat’s work, and other results (inducing in stages, etc.) will be given elsewhere.

**References**


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