COEFFICIENTS FOR ALPHA-CONVEX UNIVALENT FUNCTIONS

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Let $\alpha$ be a nonnegative real number, and let $M(\alpha)$ denote the class of normalized $\alpha$-convex univalent functions $f$ in the open unit disc $E = \{z: |z| < 1\}$, i.e., $f \in M(\alpha)$ if and only if $f$ is regular in $E$, $f(0) = f'(0) - 1 = 0$, $f(z)f'(z)/z \neq 0$ for $z \in E$, and

$$\text{Re} \left( (1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right) > 0$$

for $z \in E$ [3], [4]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, the coefficient bounds for $|a_2|$ and $|a_3|$ are known [2], [4]; an inequality relating the coefficients $|a_n|$ for $n = 2, 3, \ldots$ is found in [2]; yet the determination of the coefficient bound for $|a_n|$ has so far been an open problem.

Here we announce the general result for this coefficient problem; the proof will be published elsewhere.

**Theorem.** Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in M(\alpha)$. Let $S(n)$ be the set of all $n$-tuples $(r_1, r_2, \ldots, r_n)$ of nonnegative integers for which $r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n$, and for each such $n$-tuple define $m$ by $r_1 + r_2 + \cdots + r_n = m$. If $\gamma(\alpha, m) = \alpha(\alpha-1)(\alpha-2) \cdots (\alpha-m)$ with $\gamma(\alpha, 0) = 1$, then for $n = 1, 2, \ldots$

$$|a_{n+1}| \leq \sum_{r_1 + r_2 + \cdots + r_n = m} \frac{\gamma(\alpha, m-1) c_1 c_2^2 \cdots c_n^r}{r! r_2! \cdots r_n!},$$

where summation is taken over all $n$-tuples in $S(n)$, and

$$c_n = \frac{2(2 + \alpha)(2 + 2\alpha) \cdots [2 + (n - 1)\alpha]}{n! \alpha^n (1 + n\alpha)}.$$

The bounds in (1) are sharp and for $\alpha > 0$ attained by

$$f(z) = \left[ \frac{1}{\alpha} \int_0^z \xi^{1/\alpha - 1}(1 - \xi)^{-2/\alpha} d\xi \right]^\alpha.$$
For $\alpha=0$, we find from (1) that $|a_n| \leq n$ for $n=2, 3, \cdots$ and the bounds are attained by the function $f(z) = z(1-z)^{-2}$. For $\alpha=1$, $|a_n| \leq 1$ for $n=2, 3, \cdots$ the bounds being attained by $f(z) = z(1-z)^{-1}$.

The technique used by Goodman in [1] has been employed to get the bounds in (1) in the compact form.

Thus, it is easy to find from (1) that, e.g.,

$$|a_2| \leq 2/(1 + \alpha),$$

$$|a_3| \leq (3 + 8\alpha + \alpha^2)/(1 + \alpha)^2(1 + 2\alpha),$$

$$|a_4| \leq 4(3 + 19\alpha + 38\alpha^2 + 11\alpha^3 + \alpha^4)/3(1 + \alpha)^3(1 + 2\alpha)(1 + 3\alpha),$$

$$|a_5| \leq \frac{30 + 394\alpha + 2024\alpha^2 + 5284\alpha^3 + 6386\alpha^4 + 2638\alpha^5 + 488\alpha^6 + 36\alpha^7}{6(1 + \alpha)^4(1 + 2\alpha)^2(1 + 3\alpha)(1 + 4\alpha)}.$$

The formula (1) is however readily computable.

It may be noted that $\sup|a_{n+1}| < \sup|a_n|$ for $\alpha \geq 2$, $n=2, 3, \cdots$. Also, for a given $n$, $n=2, 3, \cdots$, $\sup|a_n|$ is a decreasing function of $\alpha$, $\alpha \geq 0$.

REFERENCES

2. P. K. Kulshrestha, Coefficient problems for a class of Mocanu-Bazilevič functions (submitted).

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