GENERALIZED SUPER-PARABOLIC FUNCTIONS

BY NEIL EKLUND

Communicated by Alberto Calderón, September 22, 1973

The purpose of this note is to announce results which generalize potential theory (superharmonic functions) to a broad class of parabolic operators. Many of the properties of superharmonic functions carry over to functions in this new class. Let \( Q = \Omega \times (0, T) \) where \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( T > 0 \) is a scalar. All functions will be defined on \( \overline{Q} \) and will be written as functions of \((x, t)\) with \( x \in \overline{\Omega} \) and \( t \in [0, T] \).

For \((x, t) \in \overline{Q}\) assume

(a) \( a_{ij}(x, t) \) is a bounded, measurable function for \( i, j = 1, 2, \ldots, n \) and assume there is a constant \( \lambda > 0 \) such that

\[
\sum a_{ij}(x, t)z_i z_j \geq \lambda |z|^2
\]

for all \( z \in \mathbb{R}^n \) and almost all \((x, t) \in Q\).

(b) \( c(x, t) \in L^p[0, T; L^q(\Omega)] \) for \( n/2p + 1/q < 1, 1 < p, q \leq \infty \).

(c) \( b_j(x, t), d_j(x, t) \in L^p[0, T; L^q(\Omega)] \) for \( j = 1, \ldots, n \) and \( n/2p + 1/q < \frac{1}{2}, 2 < p, q \leq \infty \).

The parabolic operator under consideration is defined by

\[
Lu = u_t - \{a_{ij}(x, t)u_{,i,j} + d_j(x, t)u_{,j} - b_j(x, t)u_{,j} - c(x, t)u\}
\]

where \( u_{,i} = \partial u / \partial x_i \) and an index \( i \) or \( j \) is summed over \( 1 \leq i, j \leq n \) whenever it is repeated in a product.

**Definition 1.** \( u(x, t) \) is a weak solution of \( Lu = 0 \) in \( Q \) if \( u \) is locally in \( L^2[0, T; H^{1,2}(\Omega)] \) and \( \int_Q [a_{ij}u_{,i,j} + d_j u_{,j} - b_j u_{,j} - c u - u_t] \, dx \, dt = 0 \) for all \( \phi \in C^1(\overline{Q}) \).

Let \( \partial_p Q = \{\partial Q \times [0, T]\} \cup \{\Omega \times (0)\} \) denote the parabolic boundary of \( Q \). Due to the number of definitions and results, they are stated below with no proofs.

**Theorem 1.** Let \( f \in C(\partial_p Q) \) and let \( u = u(x, t) \) be the weak solution of the boundary value problem

\[
Lu = 0 \quad \text{on} \quad Q, \quad u = f \quad \text{on} \quad \partial_p Q.
\]

Then, to each \((x, t) \in Q\), there corresponds a nonnegative Borel measure
\[ \mu_{(z,t)} \text{ on } \partial_p Q \text{ such that} \]
\[ u(x, t) = \int_{\partial_p Q} f \, d\mu_{(z,t)} \text{ on } Q. \]

In the future, write \( L(f; (x, t), Q) = \int_{\partial_p Q} f \, d\mu_{(z,t)}. \)

**Definition 2.** \( u \in S_Q \) if and only if \( u \in L^2[0, T; H^1(Q)] \) and for all \( \phi \in C_0^\infty(Q) \) with \( \phi \geq 0, \int_Q [a_{ij} \partial_{ij} \phi + b_i \phi \partial_i u - cu \phi - u \phi] \, dx \, dt \geq 0. \)

**Definition 3.** \( R_a(x_0, t_0) = \{(x, t); |x_i - x_{0_i}| < a, t_0 - a^2 < t \leq t_0\} \) is called a standard rectangle based at \((x_0, t_0).\)

**Definition 4.** \( u \in l(Q) \) if and only if
(i) \( u \neq \infty \) on \( Q, \)
(ii) \( u > -\infty \) on \( Q, \)
(iii) \( u \) is lower semicontinuous on \( Q. \)

**Definition 5.** The extended real valued Borel measurable function \( u \)
defined on an open set \( D \) is
(a) super-mean-valued at \( z \in D \) if \( L(u; z, R_\delta) \) is defined and \( u(z) \geq L(u; z, R_\delta) \) for almost all \( \delta \) with \( \delta(z) \subseteq D; \)
(b) super-mean-valued on \( D \) if it is super-mean-valued at each \( z \in D; \)
(c) locally super-mean-valued at \( z \in D \) if there is a \( \delta(z) > 0 \) such that \( \delta(z) \subseteq D \) and \( u(z) \geq L(u; z, R_\delta) \) for all \( \delta \ll \delta(z); \)
(d) locally super-mean-valued on \( D \) if it is locally super-mean-valued at each \( z \in D. \)

**Definition 6.** \( S'_Q = \{u \in l(Q); u \) is super-mean-valued on \( Q\}. \]
\( S''_Q = \{u \in l(Q); \) for any cylinder \( W = C \times (a, b) \) with \( W \subseteq Q, \) and any \( v \) with \( v \in C(W), L_v = 0 \) on \( W, \) and \( u \geq v \) on \( \partial_p W, \) it follows that \( u \geq v \) on \( W\}. \]
\( S''_Q = \{u \in l(Q); u \) is locally super-mean-valued on \( D\}. \)

**Theorem 2.** \( u \in S_Q \) with \( u \geq 0 \) on \( \partial_p Q \) implies \( u \geq 0 \) on \( Q. \)

**Corollary.** If \( c + \{d_i\}_i \leq 0 \) weakly on \( Q, \) then the weak solution \( u \) of \( Lu = 0 \) in \( Q, u = 1 \) on \( \partial_p Q \) satisfies \( 0 \leq u(x, t) \leq 1 \) on \( Q. \)

From now on assume \( c + \{d_i\}_i \leq 0 \) weakly on \( Q. \)

**Theorem 3.** Let \( u \in S''_Q. \) If, for some \( (x_0, t_0) \in Q, u(x_0, t_0) = \inf Q \leq 0, \) then \( u(x, t) = u(x_0, t_0) \) on \( \Omega \times (0, t_0). \)

**Theorem 4.** \( S_Q \subseteq S''_Q = S''_Q. \)

**Theorem 5.** Let \( F(x) \) be convex on \( E^n \) with \( F(0) \leq 0. \) If \( Lu = 0 \) on \( Q, \) then \( -F(u) \in S'_Q. \)

**Theorem 6.** Let \( F(x) \) be nondecreasing and convex on \( E^n \) with \( F(0) \leq 0. \) If \( -u \in S'_Q, \) then \( -F(u) \in S'_Q. \)
Theorem 7. If \( u \in S^' Q \), and if \( u(x, t) \geq 0 \), then there exist \( t_0, t_1 \) with \( 0 \leq t_0 \leq t_1 \leq T \) such that

\[
\begin{align*}
 u(x, t) & = 0 \quad \text{on } \Omega \times (0, t_0), \\
 0 < u(x, t) < +\infty & \text{on } \Omega \times (t_0, t_1), \\
 u(x, t) & = +\infty \quad \text{on } \Omega \times (t_1, T).
\end{align*}
\]

Theorem 8. If \( u, v \in S^' Q \) and \( c > 0 \), then
(i) \( cu \in S^' Q \),
(ii) \( u + v \in S^' Q \),
(iii) \( \inf(u, v) \in S^' Q \).

Theorem 9. \( u, -u \in S^' Q \) implies \( Lu = 0 \) weakly on \( Q \).

Theorem 10. Let \( u \in S^' Q \) and let \( R \) be a standard rectangle with \( \bar{R} \subset Q \). Set

\[
\begin{align*}
v(x, t) & = L(u; (x, t), R) \quad (x, t) \in R, \\
& = u(x, t) \quad (x, t) \in Q - R.
\end{align*}
\]

Then \( u \geq v \) on \( Q \), \( Lv = 0 \) on \( R \), and \( v \in S^' Q \).

References