MINIMAL TOTAL ABSOLUTE CURVATURE FOR ORIENTABLE SURFACES WITH BOUNDARY

BY JAMES H. WHITE

Communicated by S. S. Chern, September 22, 1973

Let \( M \) be an orientable surface with single smooth boundary curve \( C \) which is \( C^2 \) imbedded in Euclidean three-space \( E^3 \). (\( M \) may be thought of as a closed orientable surface with a single disc removed.) Let \( M_\epsilon \) be the set of points of \( E^3 \) at a distance \( \epsilon \) from \( M \). \( M_\epsilon \) is, of course, for small \( \epsilon \), an imbedded closed surface which is almost everywhere \( C^2 \). Using N. Grossman's [1] adaptation of N. Kuiper's [2] definition, we say that \( M \) has minimal total absolute curvature if \( M_\epsilon \) is tightly imbedded or has the two piece property, TPP [2].

We announce the following result:

**Theorem.** Let \( M \) be an orientable surface of genus \( g \) with a single smooth boundary curve which is \( C^2 \) imbedded in \( E^3 \). Then \( M \) has minimal total absolute curvature if and only if \( M \) has \( g = 0 \) and is a planar disc bounded by a convex curve.

The proof uses a series of integral equations and geometric arguments. The outline is as follows. First, in his paper [1], N. Grossman shows that an orientable surface \( M \) of genus \( g \) with boundary curve \( C \) has minimal total absolute curvature only if the following integral equality holds:

\[
\frac{1}{2\pi} \int_M |K| \, dA + \frac{1}{2\pi} \int_C \kappa \, ds = 1 + 2g,
\]

where \( K \) is the Gauss curvature of \( M \) and \( \kappa \) is the Frenet curvature of the boundary curve \( C \) considered as a space curve in \( E^3 \), where \( dA \) is the area element of \( M \) and \( ds \) is the arc element of \( C \). Note that the right-hand side is the sum of the betti-numbers of \( M \) and compare with Kuiper [2] for closed surfaces.

Next, the theorem of Gauss-Bonnet yields

\[
\frac{1}{2\pi} \int_M K \, dA + \frac{1}{2\pi} \int_C \kappa \, ds = 1 - 2g,
\]


Copyright © American Mathematical Society 1974
where $\kappa_g$ is the geodesic curvature of $C$ considered as a curve on the surface $M$.

Adding (1) and (2), we obtain that if $M$ has minimal total absolute curvature,

$$(3) \quad \frac{1}{2\pi} \int_{M'\{K>0\}} K \, dA + \frac{1}{2\pi} \int_C (\kappa + \kappa_g) \, ds = 2,$$

where the first integral is taken over the points of $M$ where $K > 0$.

**Lemma 1.** If $M$ has minimal total absolute curvature, then $M$ has TPP.

In [3], L. Rodriguez shows that, if $M$ has TPP,

$$(4) \quad \frac{1}{2\pi} \int_{M'\{K>0\}} K \, dA + \frac{1}{2\pi} \int_C (\kappa + \kappa_g) \, ds = 2.$$

Subtracting (4) from (3), we obtain $(1/2\pi) \int_{M'\{K>0\}} K \, dA = 0$, and hence $K \leq 0$ in the interior of $M$.

**Lemma 2.** $K \leq 0$ in the interior of $M$.

**Lemma 3.** $C$ is a plane convex curve.

Lemma 3 is proved by using Morse theory and studying the convex hull of $M$.

**Lemma 4.** $K \equiv 0$ in the interior of $M$.

This follows immediately from Lemmas 2 and 3.

Now Lemma 4 implies $\int_M |K| \, dA = 0$, and Lemma 3 implies $(1/2\pi) \int_C \kappa \, ds = 1$. Thus, in order for equation (1) to hold $g$ must be zero and $M$ must be a planar disc bounded by a convex curve.

**References**

3. L. Rodriguez, *The two-piece-property and relative tightness for surfaces with boundary* (xeroxed thesis), Brown University, Providence, R.I.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024