ADDENDUM TO: "ON EXTENSIONS OF FUNDAMENTAL GROUPS OF SURFACES AND RELATED GROUPS"

BY HEINER ZIESCHANG

Communicated by F. W. Gehring, October 4, 1973

Copying the methods of J. Nielsen [1] Theorem 1 of [2] can be proved, i.e. that a finite torsionfree extension of the fundamental group of a surface is isomorphic to the fundamental group of a surface. Indeed, the following slightly more general theorem can be proved, but it is considerably weaker than Theorem 1' of [2].

**Theorem.** Let $\mathcal{G}$ be the fundamental group of a surface $S$ and let $\mathcal{G}$ be finitely generated. Let $\mathcal{G}$ be a group which contains $\mathcal{G}$ as a normal subgroup of finite index and which has the following properties:

(i) For each $g \in \mathcal{G}$ the automorphism of $\mathcal{G}$ defined by $x \mapsto g^{-1}xg$ is induced by a homeomorphism of $S$.

(ii) If $g \in \mathcal{G}$ and $g^{-1}xg = x$ holds for all $x \in \mathcal{G}$, then $g \in \mathcal{G}$.

(iii) If $x^a = y^b = (xy)^c = 1$ holds for $x, y \in \mathcal{G}$ and $a, b, c \geq 2$, then $x, y$ generate a cyclic subgroup of $\mathcal{G}$.

Then $\mathcal{G}$ is isomorphic to a finitely generated discontinuous group of motions of the hyperbolic or euclidean plane.

I shall briefly sketch a proof of the Theorem which generalizes [1]. Let $S$ be an orientable surface with finite genus and a finite number of holes and without boundary. We consider $S$ as a Riemann surface. If the universal cover is holomorphically equivalent to the euclidean plane, everything can be proved in a similar way as in [2, Theorem 3]. Therefore we may assume that the universal cover is the hyperbolic plane $H$ which we represent by the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and the Poincaré model. The fundamental group of $S$ acts on $H$ as a group $\mathcal{G}$ of conformal transformations. We may assume that $\mathcal{G}$ contains only hyperbolic transformations except the identity. Then the methods of [1] can be applied: Each cyclic subgroup of $\mathcal{G}$ consists of motions with the same axis, and a maximal cyclic subgroup contains all elements preserving an axis. Therefore each automorphism of $\mathcal{G}$ induces a permutation of the axes of $\mathcal{G}$ and
of their base points, which lie on \( \partial H = \{ z \in \mathbb{C} \mid |z| = 1 \} \). If the automorphism is induced by a homeomorphism of \( S \) (which corresponds to a \( \mathfrak{G} \)-invariant homeomorphism of \( H \)) the mapping of the set of base points can be extended to a homeomorphism of \( \partial H \) (this extends the homeomorphism of \( H \) to a homeomorphism of the closed unit disk). So \( \mathfrak{G} \) defines a group of permutations of the axes of \( \mathfrak{G} \). For \( g \in \mathfrak{G} \) and an axis \( A \), denote by \( gA \) the image axis. An axis \( A \) is simple, if \( gA \cap A \neq \emptyset \) for \( g \in \mathfrak{G} \) implies \( gA = A \). We may restrict ourselves to the case where the elements of \( \mathfrak{G} \) are induced by orientation preserving homeomorphisms of \( S \). Now we can repeat the arguments of [1, pp. 51-78], in this more general situation and we obtain

**Lemma 1.** If \( \mathfrak{G} \) admits a simple axis, then \( \mathfrak{G} \) is isomorphic to a finitely generated discontinuous group of motions of the hyperbolic plane \( H \).

The criterion for the existence of a simple axis is the same as that in Nielsen [1, pp. 78–94]:

**Lemma 2.** \( \mathfrak{G} \) admits a simple axis, if (iii) holds.

**Remark.** If the group \( \mathfrak{G} \) contains elements \( x, y \) with \( x^a = y^b = (xy)^c = 1 \), \( a, b, c \geq 2 \), which generate a noncyclic subgroup \( \mathcal{U} \), then it must be proved that the above relations are defining relations for \( \mathcal{U} \) (which I could not obtain in all cases) and that the index of \( \mathcal{U} \) in \( \mathfrak{G} \) is “small”. The conclusion in [1, pp. 99, lines 10–21], does not seem correct to me.

**References**


**Ruhr-Universität Bochum, Institut fuer Mathematik, 463 Bochum, Postfach 2148, Germany**