THE FATOU-ZYGMUND PROPERTY FOR SIDON SETS

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A subset $X$ of a discrete abelian group $G$ is said to be a Sidon set if every bounded complex-valued function on $X$ is the restriction to $X$ of a Fourier-Stieltjes transform on $G$. In this article we give an affirmative answer to a question of J.-E. Bjork [1] and N. Th. Varopoulos [6].

THEOREM 1. Let $X$ be a symmetric Sidon subset of $G$ not containing $0_G$. Then every bounded hermitian function on $X$ is the restriction to $X$ of a positive-definite function on $G$.

In the terminology of Edwards, Hewitt and Ross [2], the set $X$ has the Fatou-Zygmund property. We refer the reader to this article and to Ross [7] for a deeper understanding of the content of Theorem 1. The proof of Theorem 1 uses the technique of [3] but the presentation we give is akin to that of [4]. Unexplained notations and definitions may be found in [5].

For technical reasons we should like $X$ to be a finite set. Thus we shall actually prove the following result.

THEOREM 2. For all $\alpha$, $0 < \alpha \leq 1$ there is a constant $C(\alpha)$ such that for every finite symmetric Sidon ($\alpha$) subset $X$ of $G$ not containing $0_G$ and every hermitian function $\phi$ on $X$ with $\|\phi\|_{\infty} \leq 1$, there exists $\mu$ a positive measure on $G$ with $\|\mu\|_{M} \leq C(\alpha)$ such that $\hat{\mu}|_{X} = \hat{\phi}$.

It is an easy consequence of Theorem 2 that the analogous statement with the word finite deleted holds. Thus Theorem 1 follows from Theorem 2. From now on let $X$ be as in Theorem 2.

We fix $n$ to be an even integer greater than or equal to four and define $\Omega$ to be the finite group of hermitian mappings from $X$ to the complex $n$th roots of unity under pointwise multiplication. If $U$ denotes the set of...
all hermitian functions of $X$ into the closed unit disc we have

\[
(*) \quad U \subseteq \sec(\pi/n) \cdot \co(\Omega)
\]

where $\co(\Omega)$ denotes the real-affine convex hull of $\Omega$. This is not true if $n=2$ or if $n$ is odd and $X$ contains elements of order two.

The next lemma is a modification of the convolution device lemma of [4].

**Lemma 3.** There exist functions $g$, $g^*$, $g^+$ and $g^-$ on $G \times \Omega$ having the following properties

1. $g=g^+-g^-$, $g^*=g^++g^-$,
2. $g^+\omega$ is positive definite on $G \forall \omega \in \Omega$,
3. $g(x, \omega) = \omega(x) \forall \omega \in \Omega, \forall x \in X$,
4. $\|g^+\|_{B(\Omega)} \leq \alpha^{-2} \forall \omega \in \Omega$,
5. $\|g^*\|_{A(\Omega)} \leq \alpha^{-2} \forall x \in G$.

**Proof.** Since $X$ is Sidon (a) there exist functions $f_\omega$ ($\omega \in \Omega$) on $G$ such that $f_\omega(x) = \omega(x) \forall \omega \in \Omega, \forall x \in X;$ $\|f_\omega\|_{B(\Omega)} \leq \alpha^{-1} \forall \omega \in \Omega$. We may assume that each $f_\omega$ is hermitian on $G$ for if not it suffices to throw away its skew-hermitian part. Thus we may write $f_\omega = f^+_\omega - f^-_\omega$ where $f^\pm_\omega$ is positive definite on $G$. Now define

\[
g^{\pm}(x, \omega) = \int f^{\pm}(x, \omega\lambda^{-1})f^{\pm}(x, \lambda) \, d\eta(\lambda)
\]

where $\eta$ is the invariant probability measure on $\Omega$. We set $g^+=g^++g^-$, $g^-=g^+-g^+$, $g=g^+-g^-$ and $g^*=g^++g^-$. Conditions (1)–(3) are easily checked and (4)–(5) follow as in [4].

Let $H$ denote the dual group of $\Omega$, that is, the $Z(n)$-module generated by $X$ and the relations $x + (-x) = 0 \ (x \in X)$. The negation mapping on $X$ induces inversion on $\Omega$

\[
\omega(-x) = \overline{\omega(x)} = \omega^{-1}(x)
\]

which in turn induces negation on $H$. The natural injection $j$ of $X$ into $H$ given by $(j(x), \omega) = \omega(x)$ thus satisfies $j(-x) = -j(x)$. A finite subset $Y$ of a discrete abelian group $F$ is said to be symmetric $n$-independent if and only if

(a) $Y$ is symmetric.
(b) If $m: Y \to Z$ and $\sum_{y \in Y} m(y) \cdot y = 0_F$ then $m(y) - m(-y) = 0 \mod n$ for all $y \in Y$ and $m(y) = 0 \mod 2$ for all $y \in Y$ with $2y = 0_F$. It is easy to prove that the subsets $j(X)$ and $\text{graph}(j) = \{(x, j(x)); x \in X\}$ are symmetric $n$-independent in $H$ and $G \times H$ respectively.
LEMMA 4. Let $0 < \varepsilon \leq 1$ and suppose that $Y$ is a symmetric $n$-independent subset of $F$. There exist functions $p^+$, $p^-$, $p^e$ and $p^o$ on $F$ such that

1. $p^+ = p^e + p^o$, $p^- = p^e - p^o$;
2. $p^\pm$ is positive definite on $F$;
3. $p^o(y) = 1/2 \varepsilon \forall y \in Y$;
4. $||p^\pm||_{B(R)} = 1$;
5. $|p^o(y)| \leq \varepsilon^2 \forall y \in F \setminus \{0_F\}$.

The letters $e$ and $o$ stand for even and odd.

PROOF. Let $Q$ denote the quotient of $Y$ induced by the equivalence relation $y_1 \sim y_2$ if and only if either $y_1 = y_2$ or $y_1 = -y_2$. For $q \in Q$ and $\chi \in \hat{F}$ we define

$$a_q^\pm(\chi) = 1 \pm \frac{\varepsilon}{2} \sum_{y \in q} \chi(y)$$

and the cosine Riesz products $p^\pm$ are defined by

$$(p^\pm)(\chi) = \prod_{q \in Q} a_q^\pm(\chi).$$

The definition of $p^e$ and $p^o$ is given by (1'). The verification of (2'), (3') and (4') is routine—see for example [5, p. 124]. To prove (5') we establish by direct calculation that

$$p^o(z) = \sum (\frac{\varepsilon}{2})^{\text{card}(R)} C_R(z)$$

where the summation is over all even subsets $R$ of $Q$ and $C_R(z)$ is the number of partial section maps $y : R \to Y$ for which $z = \sum_{e \in R} y(q)$. The definition of symmetric $n$-independence ensures that for each fixed $z$, $C_R(z)$ is nonzero for at most one value of $R$. Thus

$$|p^o(z)| \leq \sup(\frac{\varepsilon}{2})^{\text{card}(R)} C_R(z).$$

Since $\text{card}(q) \leq 2$ for all $q$ in $Q$ it follows that $C_R(z) \leq 2^{\text{card}(R)}$. Clearly $C_q(z) = 0$ for $z \neq 0_F$. Recalling that the supremum is only over sets of even cardinality we have (5').

PROOF OF THEOREM 2. We use the notation of Lemmas 3 and 4 where $Y = \text{graph}(f)$ and $F = G \times H$. We define

$$s(x, \omega) = \int [(p^+)^*(x, \omega) g^+(x, \lambda) + (p^-)^*(x, \omega \lambda^{-1}) g^-(x, \lambda)] d\eta(\lambda)$$

where $^*$ denotes the Fourier transform in the $\Omega$, $H$ duality only. By (2) and (2'), $s_\omega$ is positive definite in $G$ for each $\omega$ in $\Omega$. By (4) and (4'),
\[
\|s_\omega\|_{B(\mathcal{G})} \leq 2\alpha^{-2} \quad \forall \omega \in \Omega. \text{ Now we rewrite } s.
\]

\[
s(x, \omega) = \int (p^\delta)^*(x, \omega \lambda^{-1}) g(x, \lambda) \, d\eta(\lambda) + \int (p^\delta)^*(x, \omega \lambda^{-1}) g^*(x, \lambda) \, d\eta(\lambda)
\]

\[
= s^\delta(x, \omega) + s^\delta(x, \omega).
\]

By (3) and (3'), \( s^\delta(x, \omega) = \frac{1}{2} \epsilon \omega(x) \quad \forall \omega \in \Omega, \forall x \in X. \) By (5), (5') and since \( \delta \notin X, |s^\delta(x, \omega)| \leq \epsilon^2 \alpha^{-2} \quad \forall \omega \in \Omega, \forall x \in X. \) Hence

\[
|s(x, \omega) - \frac{1}{2} \epsilon \omega(x)| \leq \epsilon^2 \alpha^{-2} \quad \forall \omega \in \Omega, \forall x \in X.
\]

Now by real-affine convexity and the condition (\ast) we have that for each element \( \phi \) of \( U \) there exists a positive measure \( \mu \) on \( \mathcal{G} \) such that

\[
\|\mu\|_M \leq 4\epsilon^{-1} \alpha^{-2} \sec(\pi/\delta),
\]

\[
\|\mu\|_1 - \phi\|_\infty \leq 2\epsilon \alpha^{-2} \sec(\pi/\delta).
\]

Now select \( \epsilon = \frac{1}{4} \alpha^2 \cos(\pi/\delta). \) Since \( \mu\|_X - \phi \) is again hermitian on \( X, \)

Theorem 2 follows by iteration. The constant \( C(\alpha) \) may be taken to be \( 32\alpha^{-4}. \)

REFERENCES


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