AN INEQUALITY FOR THE DISTRIBUTION OF A SUM OF CERTAIN BANACH SPACE VALUED RANDOM VARIABLES

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1. Introduction. Throughout the paper $B$ is a real separable Banach space with norm $\| \cdot \|$, and all measures on $B$ are assumed to be defined on the Borel subsets of $B$. We denote the topological dual of $B$ by $B^*$. A measure $\mu$ on $B$ is called a mean zero Gaussian measure if every continuous linear function $f$ on $B$ has a mean zero Gaussian distribution with variance $\int_B [f(x)]^2 \mu(dx)$. The bilinear function $T$ defined on $B^*$ by

$$T(f, g) = \int_B f(x)g(x) \mu(dx) \quad (f, g \in B^*)$$

is called the covariance function of $\mu$. It is well known that a mean zero Gaussian measure on $B$ is uniquely determined by its covariance function.

However, a mean zero Gaussian measure $\mu$ on $B$ is also determined by a unique subspace $H_\mu$ of $B$ which has a Hilbert space structure. The norm on $H_\mu$ will be denoted by $\| \cdot \|_\mu$ and it is known that the $B$ norm $\| \cdot \|$ is weaker than $\| \cdot \|_\mu$ on $H_\mu$. In fact, $\| \cdot \|$ is a measurable norm on $H_\mu$ in the sense of [3]. Since $\| \cdot \|$ is weaker than $\| \cdot \|_\mu$, it follows that $B^*$ can be linearly embedded into the dual of $H_\mu$, call it $H_\mu^*$, and identifying $H_\mu$ with $H_\mu^*$ in the usual way we have $B^* \subseteq H_\mu \subseteq B$. Then by the basic result in [3] the measure $\mu$ is the extension of the canonical normal distribution on $H_\mu$ to $B$. We describe this relationship by saying $\mu$ is generated by $H_\mu$. For details on these matters as well as additional references see [3] and [4].

2. The basic inequality. The norm $\| \cdot \|$ on $B$ is twice directionally differentiable on $B-\{0\}$ if for $x, y \in B, x+ty \neq 0$, we have

$$\frac{d}{dt} \|x + ty\| = D(x + ty)(y)$$

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where $D: B - \{0\} \to B^*$ is measurable from the Borel subsets of $B$ generated by the norm topology to the Borel subsets of $B^*$ generated by the weak-star topology, and

$$ (d^2/dt^2) \| x + ty \| = D^2_{x+ty}(y, y) $$

where $D^2_x$ is a bounded bilinear form on $B \times B$. We call $D^2_x$ the second directional derivative of the norm, and without loss of generality we can assume $D^2_x$ is a symmetric bilinear form. That is, if $T_x$ is a bilinear form which satisfies (2.2) then $\Lambda_x(y, z) = [T_x(y, z) + T_x(z, y)]/2$ also satisfies (2.2) and $\Lambda_x$ is symmetric. Hence in all that follows we assume $D^2_x$ is a symmetric bilinear form. Of course, if the norm is actually twice Fréchet differentiable on $B$ with second derivative at $x$ given by $\Lambda_x$, then it is well known that $\Lambda_x$ is a symmetric bilinear form on $B \times B$, and in this case $D^2_x$ would be equal to $\Lambda_x$ since symmetric bilinear forms are uniquely determined on the diagonal of $B \times B$.

If $D^2_x(y, y)$ is continuous in $x$ ($x \neq 0$) and for all $r > 0$ and $x, h \in B$ such that $\| x \| \geq r$ and $\| h \| \leq r/2$ we have

$$ |D^2_{x+h}(h, h) - D^2_x(h, h)| \leq C_r \| h \|^{2+\alpha} $$

for some fixed $\alpha > 0$ and some constant $C_r$ we say the second directional derivative is Lip($\alpha$) away from zero.

We now can state our main result.

**THEOREM 2.1.** Let $B$ denote a real separable Banach space with norm $\| \cdot \|$. Let $\| \cdot \|$ be twice directionally differentiable on $B$ with the second derivative $D^2_x$ being Lip($\alpha$) away from zero for some $\alpha > 0$ and such that $\sup_{\| x \| = 1} \| D^2_x \| < \infty$. Let $X_1, X_2, \cdots$ be independent $B$-valued random variables such that for some $\beta > 0$

$$ (2.4) \quad \sup_k E \| X_k \|^{2+\delta} < \infty, \quad EX_k = 0 \quad (k = 1, 2, \cdots) $$

and having common covariance function $T(f, g) = E(f(X_k)g(X_k))$ ($f, g \in B^*$). Then, if $T$ is the covariance function of a mean zero Gaussian measure $\mu$ on $B$, it follows for $t \geq 0$ and any $\beta > 0$ that

$$ (2.5) \quad P\left( \| X_1 + \cdots + X_n \| / \sqrt{n} \leq t \right) \leq 2\mu(x: \| x \| \geq t - \beta) + O(n^{-\min(\alpha, 3)/2}) $$

where the bounding constant is uniform in $t \geq 2\beta$.

The proof of Theorem 2.1 uses a method which is due to Trotter [7]. The application of Trotter’s method in this setting depends on a number of important relationships between $H_\mu$ and $B$ as well as some of the
nontrivial properties of Gaussian measures on $B$. The details of the proof are lengthy and will be presented in [6].

3. Applications of the basic inequality. Using the inequality of Theorem 2.1 we can obtain the central limit theorem and the law of the iterated logarithm for a sequence of $B$-valued random variables.

**Theorem 3.1.** Let $B$ and $\{X_k\}$ satisfy the conditions in Theorem 2.1, and assume $\mu$ is a Gaussian measure on $B$ with covariance function $T$. Then, if $\mu_n$ denotes the measure induced on $B$ by $(X_1 + \cdots + X_n)/\sqrt{n}$, we have $\lim_n \mu_n = \mu$ in the sense of weak convergence.

The proof of Theorem 3.1 is not difficult and the main idea is to use (2.5) to prove that for each $\varepsilon > 0$ there is a finite dimensional subspace $E$ of $B$ such that

$$\mu_n(E^\varepsilon) > 1 - \varepsilon \quad (n \geq 1).$$

Here $E^\varepsilon$ is the $\varepsilon$ neighborhood of $E$ in $B$. Since the finite dimensional distributions of the sequence $\{\mu_n\}$ converge to those of $\mu$, (3.1) is then sufficient for the conclusion of Theorem 3.1.

We now turn to the law of the iterated logarithm. $LLn$ denotes log log $n$ if $n \geq 3$ and 1 for $n = 1, 2$.

**Theorem 3.2.** Let $B$ and $\{X_k\}$ satisfy the conditions in Theorem 2.1, and assume $\mu$ is a Gaussian measure on $B$ with covariance function $T$. If $K$ is the unit ball of the Hilbert space $H_\mu$ which generates $\mu$, then

$$P \left( \lim_n \left\| \frac{X_1 + \cdots + X_n}{(2n LLn)^{1/2}} - K \right\| = 0 \right) = 1$$

and

$$P \left( C \left( \left\{ \frac{X_1 + \cdots + X_n}{(2n LLn)^{1/2}} \right\} \right) = K \right) = 1$$

where $C\{a_n\}$ denotes the cluster set of the sequence $\{a_n\}$.

It is known that $K$ is a compact subset of $B$; thus (3.2) implies that with probability one the sequence $\{(X_1 + \cdots + X_n)/(2n LLn)^{1/2}\}$ is conditionally compact in $B$.

The proofs of (3.2) and (3.3) rest heavily on the inequality (2.5) and also on some of the nontrivial properties of Gaussian measures on $B$. The details will be given in [6].

Strassen's functional form of the law of the iterated logarithm for $B$-valued random variables can also be proved in this setting using (2.5) and the techniques developed in [5] where $B$ was assumed to be a real separable Hilbert space.
4. Some spaces with smooth norm. Here we provide some examples of Banach spaces to which the above results apply. \((S, \Sigma, m)\) denotes a measure space and \(m\) is a positive measure on \((S, \Sigma)\).

**Theorem 4.1.** If \(p \geq 2\) and if for \(x \in L^p(S, \Sigma, m)\) we define \(\|x\| = \left\{ \int_S |x(s)|^p \, m(ds) \right\}^{1/p}\), then the norm \(\| \cdot \|\) has two directional derivatives and the second derivative is \(\text{Lip}(\alpha)\) away from zero with \(\alpha = 1\) for \(p = 2\) or \(p \geq 3\) and \(\alpha = p - 2\) for \(2 < p < 3\). Furthermore, \(\sup_{\|x\| = 1} \| D^2_x \| \leq 2(p - 1)\).

The results of Theorem 4.1 are suggested by those in [1], but do not seem to be immediate corollaries of [1]. Their proof, however, is rather straightforward. Furthermore, the derivatives in Theorem 4.1 are actually Fréchet derivatives.

Using Theorem 4.1 and assuming \((S, \Sigma, m)\) is a \(\sigma\)-finite measure space we see that the \(L^p\) spaces \((2 \leq p < \infty)\) satisfy the conditions used above. Thus the central limit theorem and the law of the iterated logarithm are valid in these spaces. A central limit theorem for random variables with values in an \(L^p\) space \((2 \leq p < \infty)\) was previously known and appears in [2], but the log log law for non-Gaussian random variables is new for \(p > 2\).

**Bibliography**