SOME NEW RESULTS ABOUT HAMMERSTEIN EQUATIONS

BY HAIM BRÉZIS AND FELIX E. BROWDER

Communicated by S. S. Chern, August 1, 1973

Let $\Omega$ be a $\sigma$-finite measure space. Let $K$ be a (nonlinear) monotone operator and let $(Fu)(x) = f(x, u(x))$ be a Niemytski operator. We consider the Hammerstein type equation

$$u + KFu = g.$$  

A detailed discussion and a complete bibliography about equation (1) can be found in [3]. The new feature about the results we present here is the fact that we do not assume any coercivity for $F$. When $F$ is monotone and $K$ maps $L^1(\Omega)$ into $L^\infty(\Omega)$, there is no growth restriction on $F$ either (cf. Theorem 1). The monotonicity of $F$ can be weakened when $K$ is compact (cf. Theorem 4). Also some of these results are valid for systems in the case where $F$ is the gradient of a convex function (cf. Theorem 5).

Assume

(2) $K$ is a monotone hemicontinuous mapping from $L^1(\Omega)$ into $L^\infty(\Omega)$ which maps bounded sets into bounded sets,

(3) $f(x, r): \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in $r$ for a.e. $x \in \Omega$, and is integrable in $x$ for all $r \in \mathbb{R}$.

THEOREM 1. Under the assumptions (2) and (3), equation (1) has one and only one solution $u \in L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$.

Uniqueness. Let $u_1$ and $u_2$ be two solutions of (1). By the monotonicity of $K$ we get

$$\int_\Omega (u_1(x) - u_2(x)) \cdot (f(x, u_1(x)) - f(x_1, u_2(x))) \, dx \leq 0$$

which implies that $f(x, u_1(x)) = f(x, u_2(x))$ a.e. on $\Omega$ and therefore by (1), $u_1 = u_2$.

In proving existence of $u$ we shall use the following

LEMMA 1. Let $X$ be a Banach space and let $K: X \to X^*$ and $F: X^* \to X$ be two monotone hemicontinuous operators. Let $\{u_n\} \subseteq X^*$, $\{v_n\} \subseteq X$ and

AMS (MOS) subject classifications (1970). Primary 47H05, 47H15, 45G05.

1 Partially supported by NSF GP-28148.

Copyright © American Mathematical Society 1974

567
\{w_n\} \subset X^* be three sequences such that

(4) \(u_n\) converges to \(u\) in \(X^*\) for the weak* topology,
(5) \(F(u_n)\) converges to \(v\) in \(X\) for the weak topology,
(6) \(v_n\) converges to \(v\) in \(X\) for the weak topology,
(7) \(Kv_n\) converges to \(g-u\) in \(X^*\) for the weak* topology,
(8) \(\langle w_n, F(u_n) \rangle - \langle Kv_n, v_n \rangle \to 0\),
(9) \(\langle g_n, F(u_n) \rangle \to \langle g, v \rangle\) where \(g_n = u_n + w_n\).

Then \(u + KFu = g\).

**Proof of Lemma 1.** We have

\[ \langle u_n - u, F(u_n) \rangle = \langle g_n - w_n - u, F(u_n) \rangle. \]

By the monotonicity of \(K\) we get

\[ \langle Kv_n, v_n \rangle \geq \langle Kv_n, v \rangle + \langle Kv, v_n - v \rangle \]

and thus

\[ \lim \inf \langle Kv_n, v_n \rangle \geq \langle g - u, v \rangle. \]

By (8) we have

\[ \lim \inf \langle w_n, F(u_n) \rangle \geq \langle g - u, v \rangle. \]

Consequently, \(\lim \sup \langle u_n - u, F(u_n) \rangle \leq 0\). Since \(F\) is pseudomonotone (cf. [1]), we conclude that \(v = Fu\) and \(\langle u_n, F(u_n) \rangle \to \langle u, v \rangle\). Also \(\langle Kv_n, v_n \rangle \to \langle g - u, v \rangle\) since \(\langle w_n, F(u_n) \rangle = \langle g_n - u_n, F(u_n) \rangle \to \langle g, v \rangle - \langle u, v \rangle\) Thus

\[ \lim \langle Kv_n, v_n - v \rangle = 0, \]

and again, since \(K\) is pseudomonotone, we conclude that \(g - u = Kv = KFu\).

**Proof of Theorem 1.** By a shift we can always assume that \(f(x, 0) = 0\) and that \(K0 = 0\) (note that (1) can be written as \(u + \tilde{K}\tilde{F}u = \tilde{g}\), where \(\tilde{F}v = Fv - F0\), \(\tilde{K}v = K(v + F(0)) -KF0\) and \(\tilde{g} = g - KF0\)). Let \(\Omega_n\) be an increasing sequence of finite measure subsets of \(\Omega\) such that \(\bigcup_n \Omega_n = \Omega\). Let \(\chi_n\) be the characteristic function of \(\Omega_n\). Let \(F_n\) be \(F\) truncated by \(n\), i.e.,

\[ f_n(x, r) = f(x, r) \quad \text{whenever} \ |f(x, r)| < n, \]

\[ = nf(x, r) |f(x, r)| \quad \text{whenever} \ |f(x, r)| \geq n. \]

The equation

(10) \[ u_n + \chi_nK\chi_nF_n(u_n) = \chi_ng \]

has a solution.

Indeed the mapping \(K_n: v \mapsto \chi_nK\chi_nv\) is monotone hemicontinuous from \(L^2(\Omega)\) into itself.

On the other hand, the (multivalued) operator \(A\) defined on \(L^2(\Omega)\) by

\[ Av = \{w \in L^2(\Omega); v(x) = \chi_n(x)f_n(x, w(x)) \text{ a.e. on } \Omega\} \]
is maximal monotone in $L^2(\Omega)$ and $D(A)$ is bounded in $L^2(\Omega)$ ($|v|_{L^2} \leq n(\text{meas } \Omega)^{1/2}$, $v \in D(A)$). Consequently, $R(A + K_n) = L^2(\Omega)$ (cf. [2]) and (10) has a solution.

Multiplying (10) through by $F_n(u_n)$ and using the monotonicity of $K$ we get

$$\int_{\Omega} u_n \cdot F_n(u_n) \, dx \leq \int_{\Omega} |g_n F_n(u_n)| \, dx.$$  \hfill (11)

Let $C = 2 \|g\|_{L^\infty}$; we have

$$\int_{\Omega} u_n F_n(u_n) \, dx = \left[ \int_{|u_n| \geq C} u_n F_n(u_n) \, dx + \int_{|u_n| < C} u_n F_n(u_n) \, dx \right] \leq C \int_{|u_n| \geq C} |F_n(u_n)| \, dx - C \int_{|u_n| < C} |F_n(u_n)| \, dx \leq C \int_{|u_n| \geq C} |F_n(u_n)| \, dx - 2C \int_{|u_n| < C} |F_n(u_n)| \, dx.$$

Using (11) we obtain

$$\int_{\Omega} |F_n(u_n)| \, dx \leq 4 \int_{|u_n| \leq C} |F_n(u_n)| \, dx \leq 4 \int_{|u_n| \leq C} |f(x, u_n(x))| \, dx \leq C'$$

by assumption (3).

Going back to (10), we conclude that $\{u_n\}$ remains bounded in $L^\infty(\Omega)$. Therefore, by assumption (3), there is some function $h \in L^1(\Omega)$ such that

$$|F_n(u_n)(x)| \leq |f(x, u_n(x))| \leq h(x) \quad \text{a.e. on } \Omega.$$  \hfill (12)

We apply now Lemma 1 with $v_n = \chi_n F_n(u_n)$, $w_n = \chi_n K v_n$, $g_n = \chi_n g$. By extracting a subsequence, we can always assume that $u_n$ converges to $u$ weak* in $L^\infty(\Omega)$,

$F(u_n)$ converges to $v$ weakly in $L^1(\Omega)$,

$v_n$ converges to $v$ weakly in $L^2(\Omega)$,

$g_n$ converges to $g$ weak* in $L^\infty(\Omega)$.

Hence

$w_n$ converges to $g - u$ weak* in $L^\infty(\Omega)$,

$K v_n$ converges to $g - u$ weak* in $L^\infty(\Omega)$.

It remains to verify (8) and (9). We have

$$\langle w_n, F(u_n) \rangle = \int_{\Omega} \chi_n K v_n \cdot F(u_n) \, dx = \int_{\Omega} K v_n \chi_n F(u_n) \, dx$$

$$= \int_{\Omega} K v_n \cdot v_n \, dx + \int_{\Omega} \chi_n K v_n (F(u_n) - F_n(u_n)) \, dx.$$
The last term can be bounded by
\[ C \int_{\|F(u_n)\| > n} |F(u_n)| \, dx \leq C \int_{|h| > n} |h(x)| \, dx \]
which tends to zero as \( n \to +\infty \) and (8) follows.

Finally (9) holds since
\[ \langle g_n, F(u_n) \rangle = \int g_n F(u_n) \, dx = \int g F(u_n) \, dx + \int (\chi_n - 1) g F(u_n) \, dx, \]
and the last term goes to zero by Lebesgue’s theorem.

**Theorem 2 (Continuous Dependence).** Under the assumptions (2) and (3), \((I+KF)^{-1}\) is strongly continuous from \( L^\infty(\Omega) \) into \( L^1(\Omega) \) and \((I+KF)^{-1}\) is demicontinuous (from \( L^\infty(\Omega) \) strong into \( L^\infty(\Omega) \) weak*). If in addition \( K \) is strongly continuous from \( L^1(\Omega) \) into \( L^\infty(\Omega) \), then \((I+KF)^{-1}\) is strongly continuous from \( L^\infty(\Omega) \) into \( L^\infty(\Omega) \).

**Proof.** We shall prove a slightly stronger result. Let \( g_n \) be a bounded sequence in \( L^\infty(\Omega) \) such that \( g_n \to g \) a.e. on \( \Omega \). Let \( u_n = (I+KF)^{-1} g_n \) and let \( u = (I+KF)^{-1} g \). We are going to show that \( F(u_n) \to F(u) \) in \( L^1(\Omega) \).

We know, from the proof of Theorem 1, that \( \{u_n\} \) is bounded in \( L^\infty(\Omega) \) and there is some \( h \in L^1(\Omega) \) such that \( |F(u_n)| \leq h \) a.e. on \( \Omega \). Since
\[ \int (u_n - u)(F(u_n) - F(u)) \, dx \leq \int (g_n - g)(F(u_n) - F(u)) \, dx \]
and the right hand side goes to zero by Lebesgue’s theorem, we can extract a subsequence such that
\[ (u_{n_k} - u)(F(u_{n_k}) - F(u)) \to 0 \quad \text{a.e. on } \Omega. \]
Consequently, \( F(u_{n_k}) \to F(u) \) a.e. on \( \Omega \) and hence \( F(u_{n_k}) \to F(u) \) in \( L^1(\Omega) \).

By the uniqueness of the limit we conclude that \( F(u_n) \to F(u) \) in \( L^1(\Omega) \).

Using similar arguments, we can prove some variants of Theorem 1.

**Theorem 3.** Assume \( K \) is monotone, hemi-continuous and bounded from \( L^q(\Omega) \) into \( L^p(\Omega) \). Assume \( f(x, r): \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous and nonincreasing in \( r \) for a.e. \( x \in \Omega \) and is measurable in \( x \) for all \( x \in \mathbb{R} \), and satisfies
\[ |f(x, r)| \leq c(x) + c_0 |r|^{p-1} \quad \text{a.e. } x \in \Omega, \text{ for all } r \in \mathbb{R} \]
where \( c \in L^q(\Omega) \).

Then (1) has a unique solution \( u \in L^p(\Omega) \) for every \( g \in L^p(\Omega) \).

**Theorem 4.** Assume \( K \) is monotone, hemi-continuous from \( L^1(\Omega) \) into \( L^\infty(\Omega) \) and maps bounded sets of \( L^1(\Omega) \) into compact sets of \( L^\infty(\Omega) \).
Assume $f(x, r)$ is continuous in $r$ for a.e. $x \in \Omega$ and there exists $M$ such that
\[
(f(x, r) - f(x, 0))r \geq 0 \quad \text{for a.e. } x \in \Omega \text{ and for all } |r| \geq M.
\]
Suppose $f(x, r)$ is measurable in $x$ for all $r \in \mathbb{R}$ and for every constant $C$,
\[
\int_{|r| \leq C} |f(x, r)| \quad \text{is integrable.}
\]
Then (1) has a solution $u \in L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$.

The case of systems. Assume
\[
(13) \quad K \text{ is monotone hemicontinuous and bounded from } L^1(\Omega; \mathbb{R}^n) \text{ into } L^\infty(\Omega; \mathbb{R}^n).
\]
\[
(14) \quad f(x, r): \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is continuous in } r \text{ for a.e. } x \in \Omega \text{ and trimonotone in } r, \text{ i.e., for a.e. } x \in \Omega \text{ and for any sequence } r_0, r_1, r_2, r_3 = r_0 \text{ we have}
\]
\[
\sum_{i=1}^{3} (f(x, r_i), r_i - r_{i-1}) \geq 0
\]
(for example, the gradient of a convex function is trimonotone, see [4]).
\[
(15) \quad f(x, r) \text{ is measurable in } x \text{ for all } r \in \mathbb{R} \text{ and for every constant } C
\]
\[
\int_{|r| \leq C} |f(x, r)| \quad \text{is integrable.}
\]

**Theorem 5.** Under the assumptions (13), (14), (15), equation (1) has a unique solution $u \in L^\infty(\Omega; \mathbb{R}^n)$ for every $g \in L^\infty(\Omega; \mathbb{R}^n)$.

In order to bound $Fu$ in $L^1$, we use the following

**Lemma 2.** Assume (14) and (15) hold. Then for any constant $\rho > 0$, there exists $h_\rho \in L^1(\Omega)$ such that
\[
\rho |f(x, r)| \leq (f(x, r) - f(x, 0), r) + h_\rho(x) \quad \text{for a.e. } x \in \Omega, \text{ all } r \in \mathbb{R}^n.
\]
Uniqueness follows from the following

**Lemma 3.** Assume $B$ is continuous and trimonotone from a Hilbert space $H$ into itself. Let $u, v \in H$ be such that
\[
(Bu - Bv, u - v) = 0.
\]
Then $Bu = Bv$.

Along the same lines one can prove the following lemma which leads to stability results.

**Lemma 4.** Assume $B$ is trimonotone and Hölder continuous with exponent $\alpha \leq 1$ (i.e., $|Bu - Bv| \leq L|u - v|^\alpha$ for all $u, v \in H$).
Then there exists a constant $k > 0$ such that

$$(Bu - Bv, u - v) \geq k |Bu - Bv|^{1+1/a} \text{ for all } u, v \in H.$$