MANIFOLDS WITH THE FIXED POINT PROPERTY. I

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1. Introduction. Suppose that \( f : M \rightarrow M \) is a map of the simply connected closed (smooth or PL) manifold \( M \) which preserves a given geometric structure. We shall consider the question of when \( f \) has a fixed point. (The geometric structure is described by an element \( \xi \) in \( K_\mathbb{R}(M) \), the Grothendieck group of real vector bundles over \( M \). If \( \deg f = 1 \), then for \( f \) to preserve \( \xi \) means just that \( f^* \xi = \xi \), and the appropriate notion when \( \deg f \neq 1 \) is given below in §2. Such maps are said to be \((\xi, \lambda)\)-maps with \( \lambda \) an integer.) Since \( M \) is simply connected, one need only compute the Lefschetz number \( \mathcal{L}(f) \) of \( f \). Thus there are three natural stages to the solution: the determination of the induced homomorphism \( f^* : \mathbb{H}^*(M; \mathbb{Z}) \rightarrow \mathbb{H}^*(M; \mathbb{Z}) \) first below the middle dimension, then in the middle dimension (when \( \dim M \) is even), and finally the determination of how the two are related to each other and how they determine the behaviour above the middle dimension.

As a first step in this direction, we consider here the case of \((2m-1)\)-connected \( M \) of dimension \( 4m \) whose intersection pairing is definite (said to be of class \( \mathcal{M}_{4m} \)). It is shown that if \( \xi \) is asymmetric enough in a suitable sense (described below in §2), then any \((\xi, \lambda)\)-map \( f : M \rightarrow M \) has a fixed point. In particular it follows that if the tangent bundle \( \tau(M) \) of \( M \) is asymmetric enough, then a \((\tau M, 1)\)-map \( f : M \rightarrow M \) has a fixed point. Therefore every homeomorphism of such a manifold \( M \) has a fixed point. It is also shown that the product of \((\xi, \lambda)\)-maps with \( \xi \) being asymmetric also has a fixed point.

[Note. At this point I would like to thank Ed Fadell for the suggestions and stimulation offered in many good conversations on this topic.]

2. Statement of results. Suppose that \( M \) is a smooth (or PL) simply connected closed manifold of dimension \( 4m \). A map \( f : M \rightarrow M \) is said to be a \((\xi, \lambda)\)-map, where \( \lambda \) is an integer, if and only if \( f^* \xi = \lambda \xi + p^* \eta \) where

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\( p: M \to S^{4m} \) is a map of degree 1 and \( \eta \in K_R(S^{4m}) \). (Note that a diffeomorphism \( f: M \to M \) is a \((\tau M, 1)\)-map, \( \tau(M) \) being the tangent bundle of \( M \).)

Assume now that \( M \) is \((2m-1)\)-connected, and suppose that the intersection pairing

\[ \varphi: H^{2m}(M; \mathbb{Z}) \times H^{2m}(H; \mathbb{Z}) \to \mathbb{Z} \]

is positive definite. The class of such manifolds will be denoted by \( \mathcal{M}_{4m} \).

One can easily show that this implies that \( \deg f = \lambda^2 \) and the Lefschetz number \( \mathcal{L}(f) = 1 + s\lambda + \lambda^2 \), with \( s \) rational and \( |s| \leq \sigma \), \( \sigma \) being the signature of \( \varphi \). Hence \( \mathcal{L}(f) \neq 0 \) for \( |\lambda| > \sigma \). On the other hand, the behaviour of \( \mathcal{L}(f) \) for \( |\lambda| \leq \sigma \) is quite different, and thus \( \sigma \) is, in a sense, a critical threshold.

To describe the case \( |\lambda| \leq \sigma \), one shows first that there is a basis \( \mathcal{S} = \{x_1, \ldots, x_a\} \) for \( H_{2m}(M; \mathbb{Z}) \) with the property that \( \langle x_i, c_m \rangle = \beta_i \sigma \) where \( \beta = \min(x, c_m) \) and \( s_i \) are integers such that \( s_1 = 1 \), \( s_i - s_{i-1} > 0 \) for all \( j > i \), and \( c_m \) the \( m \)th Chern class of \( f \).

The basis \( \mathcal{S} = \{x_1, \ldots, x_a\} \) defines a critical region for \( \xi \). If \( x, y \in H_{2m}(M; \mathbb{Z}) \) and \( xy \) denotes their intersection number, then the critical region is the set

\[ B_{\mathcal{S}} = \{x \in H_{2m}(M; \mathbb{Z}) \mid x^2 \leq \sigma^2 \mu_{\mathcal{S}}\} \]

where \( \mu_{\mathcal{S}} = \max_i x_i^2 \). Now let \( \beta_{\mathcal{S}} \) be the smallest integer such that \( |a_i| < \beta_{\mathcal{S}} - \sigma \) for all \( i \), where \( \sum_i a_i x_i \in B_{\mathcal{S}} \). \( \xi \) will be said to be sufficiently asymmetric if, and only if, \( \beta \geq \beta_{\mathcal{S}} \).

**Theorem 2.1.** Suppose that \( \xi \) is sufficiently asymmetric. Then any \((\xi, \lambda)\)-map \( f: M \to M \) has a fixed point, where \( M \in \mathcal{M}_{4m} \) and \( m > 4 \).

The following is an immediate consequence.

**Theorem 2.2.** Suppose that \( M \in \mathcal{M}_{4m} \) with \( m \) even and \( m > 4 \), and assume that \( \tau(M) \), the tangent bundle of \( M \), is sufficiently asymmetric. Then any \((\tau M, 1)\)-map \( f: M \to M \) has a fixed point. In particular, any homeomorphism of \( M \) has a fixed point.

The next theorem describes the behaviour of the products of \((\xi, \lambda)\)-maps.

**Theorem 2.3.** Suppose that \( M' \) and \( M'' \) are two manifolds in \( \mathcal{M}_{4m} \). and \( \mathcal{M}_{4m} \) with \( m' \), \( m'' > 4 \) Let \( \xi' \in K_R(M') \) and \( \xi'' \in K_R(M'') \) be sufficiently asymmetric, and put \( \xi = \xi' \otimes \xi'' \) where \( \otimes \) is the tensor product. Then any \((\xi, \lambda)\)-map \( f: M' \times M'' \to M' \times M'' \) has a fixed point.
3. Construction of $(\xi, \lambda)$-maps. In view of the preceding, it is important to know whether there is a $(\xi, \lambda)$-map $f: M \to M$. A map such as $f$ has degree $\lambda^2$, and therefore the question becomes whether there is a map $f: M \to M$ of a given degree and whether a map of a given degree preserves a given $\xi \in K_\mathbb{Z}(M)$. Let therefore $\alpha: H_{2m}(M; \mathbb{Z}) \to \pi_{2m-1}SO$ be the map which associates to $x$ the characteristic class of the induced bundle $g^*\tau(M)$, $g$ being an imbedding $S^{2m} \to M$ realizing $x$.

**Theorem 3.1.** Suppose that $\gamma: H^{2m}(M; \mathbb{Z}) \to H^{2m}(M; \mathbb{Z})$ is a monomorphism such that $\varphi(\gamma x, \gamma y) = \lambda^2 \varphi(x, y)$ for all $x, y \in H^{2m}(M; \mathbb{Z})$, where $\lambda$ is a given integer and $\varphi$ is the intersection pairing in $M$. Assume also that $\gamma(\alpha) = \lambda \alpha$. Then there is a map $f: M \to M$ such that $\gamma$ is the induced homomorphism on cohomology, provided that $J(\lambda(\lambda-1)\alpha(x)) = 0$ for all $x \in H_{2m}(M; \mathbb{Z})$, with $J$ being the $J$-homomorphism (cf. [2, Lemma 10] and [1, Theorem 5]).

Whether or not a map $f: M \to M$ of a given degree preserves a given $\xi \in K_\mathbb{Z}(M)$ is decided by considering the characteristic classes of $\xi$ and $f^*\xi$.

Thus the question of finding a $(\xi, \lambda)$-map $f: M \to M$ amounts to finding a homomorphism $\gamma: H^{2m}(M; \mathbb{Z}) \to H^{2m}(M; \mathbb{Z})$ which preserves the intersection pairing $\varphi$, the stable tangential structure $\alpha$, and the Chern class of $\xi$. If $M$ is almost parallelizable, then $\alpha$ is trivial, $\tau(M)$ has a large measure of symmetry, and the existence of $(\xi, \lambda)$-maps depends only on $\xi$ and how large the group of automorphisms of $\varphi$ is. In particular, if $\lambda = 1$ and $\xi = \tau(M)$, it follows that every quadratic automorphism $\gamma: H^{2m}(M; \mathbb{Z}) \to H^{2m}(M; \mathbb{Z})$ is induced by a corresponding homeomorphism $f: M \to M$.

**References**

1. C. T. C. Wall, Classification of $(n-1)$-connected $2n$-manifolds, Ann of Math. (2) 75 (1962), 163–189. MR 26 #3071.