ON THE BERGMAN KERNEL AND BIHOLOMORPHIC MAPPINGS OF PSEUDOCONVEX DOMAINS

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THEOREM 1. Let $D_1, D_2 \subset \mathbb{C}^n$ be strictly pseudoconvex domains with smooth boundaries and suppose that $F: D_1 \to D_2$ is biholomorphic (i.e., $F$ is an analytic homeomorphism). Then $F$ extends to a diffeomorphism of the closures, $\bar{F}: \bar{D}_1 \to \bar{D}_2$.

The main idea in proving Theorem 1 is to study the boundary behavior of geodesics in the Bergman metrics (see [2]) of $D_1$ and $D_2$. To do so, we use a rather explicit formula for the Bergman kernels of $D_1$ and $D_2$. We begin with a few definitions. Let $D = \{z \in \mathbb{C}^n | \varphi(z) > 0\}$ be a strictly pseudoconvex domain, where $\varphi \in C^\infty(\mathbb{C}^n)$ satisfies $\text{grad} \varphi \neq 0$ on $\partial D$.

1. Let $\mathcal{L}(\omega)$ denote the Levi form, i.e. the quadratic form

$$\mathcal{L}(\omega) \, dz \, \overline{dz} = \sum_{j,k} \left( \frac{\partial^2 (-\varphi)}{\partial z_j \partial \overline{z}_k} \right) |dz_j \, \overline{dz_k}|$$

restricted to the subspace $\{dz \in \mathbb{C}^n | \sum_j (\partial \varphi/\partial z_j)|_\omega \, dz_j = 0\}$ of $\mathbb{C}^n$.

2. For $\omega_1, \omega_2 \in D$, set

$$\rho(\omega_1, \omega_2) = |\omega_1 - \omega_2|^2 + |(\omega_1 - \omega_2) \cdot (\partial \varphi/\partial \omega)|_{\omega_1}.$$ (See [2] again.)

3. A smooth function $\varphi$ defined on $\bar{D} \times \bar{D}$ has weight $k$ (where $k \geq 0$ is an integer or half-integer) if the following estimate holds.

$$|\varphi(\omega_1, \omega_2)| \leq C(\varphi(\omega_1) + \varphi(\omega_2) + \rho(\omega_1, \omega_2))^k$$

4. Set

$$X(z, \omega) = \varphi(\omega) + \sum_j \frac{\partial \varphi}{\partial \omega_j} |(z_j - \omega_j) + \frac{1}{2} \sum_{i,k} \left( \frac{\partial^2 \varphi}{\partial \omega_i \partial \omega_k} \right) |(z_j - \omega_j)(z_k - \omega_k)|$$


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Elementary calculations show that \( X(z, \omega) \) has weight 1, and that
\[
|X(z, \omega)| \geq c(\psi(z) + \psi(\omega) + \rho(z, \omega)) \text{ in a region of the form } R_\delta = \{(z, \omega) \in \bar{D} \times D \mid |\psi(z) + \psi(\omega) + |z - \omega| < \delta\}.
\]

**Theorem 2.** The Bergman kernel \( K(z, \omega) \) for \( D \) has an asymptotic expansion
\[
K(z, \omega) \sim c \left| \text{grad } \psi(\omega) \right|^2 \det \mathcal{L}(\omega) X^{-(n+1)}(z, \omega) + \sum_{j=1}^{\infty} \phi_j(z, \omega) X^{-m_j}(z, \omega) + \overline{\phi}(z, \omega) \log X(z, \omega),
\]
where \( c \) is a constant, \( \phi_j \) and \( \overline{\phi} \) are smooth functions, \( \log \) denotes the principal branch of the logarithm on \( \{\text{Re}(\zeta) > 0\} \), weight \( (\phi_j) - m_j \geq -n - \frac{1}{2} \), and weight \( (\phi_j) - m_j \to \infty \) as \( j \to \infty \). The expansion (5) is valid in a region \( R_\delta \), and the symbol \( \sim \) means that for any integer \( k \),
\[
K(z, \omega) = c \left| \text{grad } \psi(\omega) \right|^2 \det \mathcal{L}(\omega) X^{-(n+1)}(z, \omega) + \sum_{j=1}^{\infty} \phi_j(z, \omega) X^{-m_j}(z, \omega) - \overline{\phi}(z, \omega) \log X(z, \omega) \in C^k(\bar{R}_\delta)
\]
for \( N \) large enough.

**Corollary.** \( K(z, z) = \Phi(z) \psi^{-(n+1)}(z) + \overline{\Phi}(z) \log \psi(z) \), where \( \Phi, \overline{\Phi} \in C^\infty(\bar{D}) \) and \( \Phi \neq 0 \) near \( \partial D \).

Although \( \overline{\Phi} \) vanishes on the unit ball, it can be nonzero, even on very smooth (say, real-analytic) domains.

The proof of Theorem 2 is based on an elementary fact.

**Lemma 1.** Given \( p \in \partial D \), we can find a region \( \bar{D} \) internally tangent to \( D \) to third order at \( p \), and an explicit biholomorphic change of co-ordinates \( F \) mapping a neighborhood of \( p \) in \( D \) to a neighborhood of \( \bar{F}(p) \) in the unit ball.

Once Lemma 1 is established, we can use \( \bar{F} \) to pull the Bergman kernel from the unit ball back to \( \bar{D} \); and since \( \bar{D} \) so closely approximates \( D \) near \( p \), we may hope that the (known) Bergman kernel for \( D \) provides a close approximation to the (unknown) Bergman kernel for \( \bar{D} \). Having thus obtained a candidate for an approximate Bergman kernel, we use a successive approximation procedure to prove (5).

Now we can attack Theorem 1 by using the corollary to Theorem 2 to make explicit differential-geometric calculations with the Bergman metric. We need two more definitions.

(6) For a fixed point \( z^0 \in D \) and a unit vector \( \omega \in S^{2n-1} \subseteq C^n \), let \( t \to \gamma(t, \omega, z^0) \) be the path of a particle moving with unit speed (in the Bergman metric) along the geodesic in \( D \) starting at \( t = 0 \) at the point
$z^0$ and travelling in the direction $\omega$. We say that $(z^0, \omega^0) \in D \times S^{2n-1}$ is pseudotransversal if the map $\omega \mapsto \pi_{z^0}(\omega) = \lim_{t \to \infty} \gamma(t, \omega, z^0)$ is well defined for $\omega$ close to $\omega^0$ in $S^{2n-1}$ and provides a diffeomorphism of a small open neighborhood of $\omega^0 \in S^{2n-1}$ onto a small open neighborhood of $\pi_{z^0}(\omega) \in \partial D$.

(7) Let $t \to \gamma(t)$ be a geodesic in $D$, and define $\omega_\gamma(t) = \text{the unit vector in the direction } d\gamma(t)/dt$. If $(\gamma(t), \omega_\gamma(t)) \in D \times S^{2n-1}$ is pseudotransversal for all $t$ larger than some fixed $T$, then we call $\gamma$ a pseudotransversal geodesic.

**Lemma 2.** (a) Every geodesic $\gamma(t)$ not remaining in a fixed compact subset of $D$ for all $t \geq 0$ is pseudotransversal.

(b) Every point $p \in \partial D$ is $\pi_{z^0}(\omega^0)$ for a certain $(z^0, \omega^0) \in D \times S^{2n-1}$.

Theorem 1 is a simple consequence of Lemma 2 and a result of Vormoor [1] which states that under the hypotheses of Theorem 1, $F$ extends to a continuous mapping $\bar{F}: \bar{D}_1 \to \bar{D}_2$. For, given $p_1 \in \partial D_1$, we use Lemma 2(b) to find a geodesic $\gamma_1(t)$ in $D_1$ with $\lim_{t \to \infty} \gamma_1(t) = p_1$. Since $F$ is an isometry of Bergman metrics, the path $\gamma_2(t) = F(\gamma_1(t))$ is a geodesic in $D_2$, and by Lemma 2(a), both $\gamma_1$ and $\gamma_2$ are pseudotransversal. Set $p_2 = \lim_{t \to \infty} \gamma_2(t)$, and pick $T$ so large that $(z_1, \omega_1) = (\gamma_1(T), \omega_{\gamma_1}(T))$ and $(z_2, \omega_2) = (\gamma_2(T), \omega_{\gamma_2}(T))$ are both pseudotransversal. Since the differential of $F$ induces a diffeomorphism $(dF)^\sim$ between the unit tangent vectors based at $z_1$ and those based at $z_2$, we have a commutative diagram

$$
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{(dF)^\sim} & S^{2n-1} \\
\pi_{z_1} \downarrow & & \downarrow \pi_{z_2} \\
\partial D_1 & \xrightarrow{\bar{F}} & \partial D_2
\end{array}
$$

where the maps $\pi_{z_1}$ and $\pi_{z_2}$ are defined in small neighborhoods of $\omega_1 = \pi_{z_1}^{-1}(p_1)$ and $\omega_2 = \pi_{z_2}^{-1}(p_2)$. All the maps in the diagram, except $\bar{F}$, are already known to be diffeomorphisms. Hence $\bar{F}$ must also be a diffeomorphism from a neighborhood of $p_1$ to a neighborhood of $p_2$, which proves Theorem 1.

**References**

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