ON VITALI-HAHN-SAKS TYPE THEOREMS
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In recent years extensive work has been done on the Vitali-Hahn-Saks theorem and its relatives. Seever [13] considered the question of extending the Vitali-Hahn-Saks theorem to the case where the domain is a Boolean algebra which is not necessarily sigma complete. Brooks and Jewett [2] established results for a strongly bounded map defined on a Boolean sigma algebra of sets with values in a Banach space. Further generalizations to group-valued set functions have been studied by the Poznán school (see [5], [6], [7], [8], [9], [11], [12]). The work of all these authors is generalized herein to the case of strongly bounded maps defined on Boolean algebras with the Seever property and taking values in a Banach space. Some applications other than those considered herein and the final generalization to the group-valued case can be found in [10].

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1. Notation and definitions. A Boolean algebra $\mathcal{B}$ has the property (I) if and only if for any sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{B}$ satisfying $x_n \leq y_m$ for all $n, m$, there exists $x \in \mathcal{B}$ such that $x_n \leq x \leq y_n$ for all $n$. This condition is equivalent to the condition: given any sequences $\{a_n\}$ and $\{b_n\}$ in $\mathcal{B}$ satisfying $a_n \wedge a_m = 0$, $b_n \wedge b_m = 0$ for $n \neq m$ and $a_n \wedge b_m = 0$ for all $n, m$, there exists an element $a$ in $\mathcal{B}$ such that $a \geq a_n$ and $b_n \wedge a = 0$ for all $n$.

Unless signified otherwise, $\mathcal{B}$ will be used in this paper to denote a Boolean algebra with the property (I). The symbol $X$ denotes a Banach space and $X^*$ its Banach space dual.

A finitely additive $\mu: \mathcal{B} \rightarrow X$ is bounded whenever there exists $M > 0$ such that $\|\mu(b)\| \leq M$ for all $b \in \mathcal{B}$; $\mu$ is said to be strongly bounded if $\|\mu(e_n)\| \rightarrow 0$ as $n \rightarrow \infty$ for each disjoint sequence $e_1, \ldots, e_n, \ldots$ of elements in $\mathcal{B}$. A sequence $\mu_n: \mathcal{B} \rightarrow X$, $n = 1, 2, \ldots$, is uniformly strongly bounded if for each disjoint sequence $\{e_n\} \subset \mathcal{B}$, $\lim_n \sup_k \|\mu_k(e_n)\| = 0$. By grouping it is easy to see that if $\mu$ is strongly bounded and $\{e_n\} \subset \mathcal{B}$ is disjoint, then $\sum_{n=1}^{\infty} \mu(e_n)$ is an unconditionally convergent series in $X$. A map $\mu: \mathcal{B} \rightarrow X$ is...
countably additive if for every disjoint sequence \( \{e_n\} \subset \mathcal{B} \) with \( \forall_n e_n \in \mathcal{B} \), the equality \( \mu(\bigvee_n e_n) = \sum_n \mu(e_n) \) holds. The semivariation of \( \mu \) on \( b \in \mathcal{B} \), denoted by \( \|\mu\|(b) \), is defined to be \( \sup(\|\mu(a)\|: a \in \mathcal{B}, a \subseteq b) \). It is easily shown that \( \mu: \mathcal{B} \rightarrow \mathcal{X} \) is strongly bounded if and only if \( \|\mu\|: \mathcal{B} \rightarrow [0, \infty) \) is strongly bounded (though \( \|\mu\| \) need not be additive).

2. Main results.

**Theorem 1.** Let \( \mu_n: \mathcal{B} \rightarrow \mathcal{X} \) be finitely additive and strongly bounded for \( n = 1, 2, \cdots \). If \( \lim_n \mu_n(e) = 0 \) for each \( e \in \mathcal{B} \), then \( \{\mu_n: n \in \mathbb{N}\} \) is uniformly strongly bounded.

**Proof.** Suppose not. Then there exists a sequence \( \{e_n\} \) of disjoint elements of \( \mathcal{B} \), a number \( \varepsilon > 0 \), and a sequence \( m_1 < m_2 < m_3 < \cdots \) of positive integers (to simplify notation, assume \( n = m \)) such that for each \( n \in \mathbb{N} \), \( \|\mu_n(e_n)\| > 4\varepsilon \).

Let \( i_1 = 1 \). Partition the set \( \mathbb{N} \setminus \{1\} \) into an infinite number of infinite disjoint sets \( \pi^1_n, n = 1, 2, 3, \cdots \). Utilizing property (i) we can choose a sequence \( f^1_n, n = 1, 2, \cdots, \) of disjoint elements in \( \mathcal{B} \) such that:

\[
\begin{align*}
(a_1) & \quad f^1_n \geq e_i \text{ for all } i \in \pi^1_n, n = 1, 2, \cdots ; \\
(b_1) & \quad f^1_n \land e_i = 0, n = 1, 2, \cdots ; \\
(c_1) & \quad f^1_n \land e_j = 0 \text{ for all } j \in (\mathbb{N} \setminus \{1\}) \setminus \bigcup_{i=1}^{\mathbb{N}} \pi^1_i.
\end{align*}
\]

As \( \|\mu_{i_1}\|(f^1_{i_1}) \rightarrow 0 \) (\( n \rightarrow \infty \)) there exists an \( n_1 \in \mathbb{N} \) such that \( \|\mu_{i_1}\|(f^1_{i_1}) < \varepsilon \).

Choose \( i_2 \in \pi^1_{n_1} \) such that \( i_2 > i_1 \) and \( \|\mu_{i_2}\|(e_{i_2})\| < \varepsilon/4 \). Partition the set \( \pi^1_{n_1} \setminus \{i_2\} \) into an infinite number of infinite disjoint sets \( \pi^2_n, n = 1, 2, \cdots \).

Again by property (i) there exists a sequence \( f^2_n, n = 1, 2, \cdots, \) of disjoint elements in \( \mathcal{B} \) such that:

\[
\begin{align*}
(a_2) & \quad f^2_n \geq e_i \text{ for all } i \in \pi^2_n, n = 1, 2, \cdots ; \\
(b_2) & \quad f^2_n \land (e_{i_1} \lor e_{i_2}) = 0, n = 1, 2, \cdots ; \\
(c_2) & \quad f^2_n \land (e_j) = 0 \text{ for all } j \in (\pi^1_{n_1} \setminus \{i_2\}) \setminus \bigcup_{i=1}^{\mathbb{N}} \pi^2_i.
\end{align*}
\]

There exists an integer \( n_2 \in \mathbb{N} \) such that \( \|\mu_{i_2}\|(f^2_{i_2}) < \varepsilon \). Choose \( i_3 \in \pi^2_{n_2} \) such that \( i_3 > i_2 \) and \( \|\mu_{i_3}\|(e_{i_3})\|, \|\mu_{i_3}\|(e_{i_2})\| < \varepsilon/8 \). Proceed in this fashion to obtain a sequence \( f^k_{i_k}, k = 1, 2, \cdots \), of elements of \( \mathcal{B} \) and a sequence \( i_k, k = 1, 2, \cdots \), of positive integers such that:

\[
\begin{align*}
(1) & \quad f_n \geq e_{i_k}, k \geq n; \\
(2) & \quad f_n \land e_{i_k} = 0, 1 \leq k \leq n; \\
(3) & \quad \|\mu_{i_k}\|(f_n) < \varepsilon, n = 1, 2, \cdots; \\
(4) & \quad \|\mu_{i_k}\|(e_{i_k})\| < \varepsilon/2^n, 1 \leq k < n; \\
(5) & \quad \|\mu_{i_k}\|(e_{i_k})\| > 4\varepsilon, n = 1, 2, \cdots.
\end{align*}
\]

Let \( h_n = f_n \lor \bigvee_{k=1}^n e_{i_k} \). Then \( h_n \geq e_{i_k} \) for all \( n, k \). Choose \( c \in \mathcal{B} \) such that

\[
(6) \quad h_n \geq c \geq e_{i_n} \quad \text{for all } n.
\]
Noticing that $\mu_{i_n}(c) = \mu_{i_n}(h_n - e_{i_n}) - \mu_{i_n}(h_n - c) + \mu_{i_n}(e_{i_n})$, we have

$$
\|\mu_{i_n}(c)\| \leq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(h_n - e_{i_n})\| - \|\mu_{i_n}(h_n - c)\|
$$

$$
= \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(f_n \vee (\bigvee_{k=1}^n e_{i_k})) \wedge e_{i_n}\|
$$

$$
- \|\mu_{i_n}(f_n \vee (\bigvee_{k=1}^n e_{i_k})) \wedge c'\|,
$$

which by (2) is

$$
\geq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(f_n \wedge e_{i_n}')\|
$$

$$
- \|\mu_{i_n}(f_n \wedge (\bigvee_{k=1}^n e_{i_k}))\| - \|\mu_{i_n}(f_n \wedge c')\| - \|\mu_{i_n}(f_n \wedge (\bigvee_{k=1}^n e_{i_k}) \wedge c')\|.
$$

Applying (2), (6) and the disjointness of the $e_{i_k}$'s yields

$$
\geq \|\mu_{i_n}(e_{i_n})\| - \|\mu_{i_n}(f_n)\|
$$

$$
- \|\mu_{i_n}(e_{i_n})\| - \cdots - \|\mu_{i_n}(e_{i_{n-1}})\| - \|\mu_{i_n}(f_n \wedge c')\|,
$$

which by (5), (3) and (4) is $> 4\varepsilon - \varepsilon - (n-1)\varepsilon / 2^n - \varepsilon \geq \varepsilon$. Since $\|\mu_{i_n}(c)\| \geq \varepsilon$ holds for infinitely many $n$, $\lim_n \mu_{i_n}(c) = 0$, a contradiction.

The proofs of some of the corollaries yielded by Theorem 1 are, for the most part, minor alterations to proofs presented elsewhere; in these cases the appropriate references are given.

**Corollary 1** [2, Corollary 1.2]. Let $\mu_n : \mathcal{B} \to X$ be finitely additive and strongly bounded for $n=1, 2, \ldots$. If $\lim_n \mu_n(e) = \mu(e)$ exists for each $e \in \mathcal{B}$, then $\mu$ is strongly bounded and the $\mu_n, n=1, 2, \ldots$, are uniformly strongly bounded.

**Corollary 2.** Let $\mu_n : \mathcal{B} \to X$ be countably additive for $n=1, 2, \ldots$. If $\lim_n \mu_n(e) = \mu(e)$ exists for each $e \in \mathcal{B}$, then $\mu$ is countably additive and the $\mu_n, n=1, 2, \ldots$, are uniformly countably additive.

**Corollary 3** [3, Theorem 1.6]. Let $X$ be any separable Banach space and let $\mu : \mathcal{B} \to X$ be bounded and finitely additive. Then $\mu$ is strongly bounded.

Another corollary is the following result proved differently by N. J. Kalton in an unpublished manuscript.

**Corollary 4.** Let $X$ be a weakly compactly generated Banach space and let $\mu : \mathcal{B} \to X$ be bounded and finitely additive. Then $\mu$ is strongly bounded.
PROOF. Let \( \{e_n\} \) be a disjoint sequence in \( \mathcal{B} \) and let \( [\mu(e_n)] = X_0 \) denote the closed linear span of \( \{\mu(e_n): n \in N\} \). Then \( X_0 \) is a separable subspace of the weakly compactly generated space \( X \); hence by a result of Amir and Lindenstrauss [1, Lemma 4], there is a separable subspace \( Y \) of \( X \) such that \( X_0 \subseteq Y \) and \( Y \) is complemented in \( X \). Suppose \( P: X \to Y \) is the projection. Then Corollary 1.2 yields \( P \circ \mu(e_n) \to 0, n \to \infty \). But \( P \circ \mu(e_n) = \mu(e_n) \) for each \( n \). Therefore, \( \mu \) is strongly bounded.

**Corollary 5** [4, Corollary 5]. Let \( \mu_n: \mathcal{B} \to X \) be strongly bounded for \( n=1, 2, \cdots \). Suppose \( \mu(e) = \text{weak-limit}_n \mu_n(e) \) exists for each \( e \in \mathcal{B} \). Then \( \mu \) is strongly bounded.

**Proof.** The boundedness of \( \mu \) follows from the Banach-Steinhaus theorem and Corollary 1.1 applied to the functions \( f \mu_n, f \mu \) where \( f \in X^* \). For each \( n \) let \( B_n = \mu_n(\mathcal{B}) \) and let \( Y \) be the closed linear span of \( \bigcup_n B_n \). By the definition of \( \mu \) and Mazur's theorem we have \( \mu(\mathcal{B}) \subseteq Y \). We claim that \( Y \) is weakly compactly generated.

For each \( n \) let \( M_n = \sup\{\|\mu_n(b)\|: b \in \mathcal{B}\} \). Let \( B = \bigcup_n B_n/(n \cdot M_n) \). The closed linear span of \( B \) is \( Y \) and \( B \) is relatively weakly compact. To see the last assertion, let \( \{y_n\} \) be a sequence in \( B \). Since each \( \mu_n \) is strongly bounded, \( B_n \), and hence \( B_n/(n \cdot M_n) \), is relatively weakly compact [14]. So if \( \{y_n\} \) returns infinitely often to one of the \( B_n/(n \cdot M_n) \)'s, we can extract a weakly convergent subsequence. If \( \{y_n\} \) does not return infinitely often to any \( B_n/(n \cdot M_n) \) then there exist strictly increasing sequences \( (m_k) \) and \( (n_k) \) of positive integers such that \( y_{m_k} \in B_{m_k}/(n_k \cdot M_{n_k}) \) for each \( k \). It follows that \( \|y_{m_k}\| \leq 1/n_k \to 0 \) as \( k \to \infty \). Thus \( \{y_n\} \) has a norm convergent subsequence.

With the proof proceeding as in [2, Theorem 3] we have the following Vitali-Hahn-Saks theorem.

**Theorem 2.** Let \( \mu_n: \mathcal{B} \to X \) be finitely additive and strongly bounded, for \( n=1, 2, \cdots \). Suppose \( v \) is a nonnegative monotone set function defined on \( \mathcal{B} \) and each \( \mu_n \ll v \). Assume that \( \lim_n \mu_n(e) \) exists for each \( e \in \mathcal{B} \). Then \( \lim_{v(e) \to 0} \|\mu_n(e)\| = 0 \) uniformly in \( n \).

**REFERENCES**


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