A LOWER ESTIMATE FOR EXPONENTIAL SUMS

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1. Introduction. In this note we present two theorems on exponential sums (see Theorems 1 and 2 below). Although seemingly unrelated, both results are motivated by the study of a certain type of lower estimates of exponential sums in the complex domain. Thus while Theorem 2 is related to the validity of this estimate for all discrete exponential sums, Theorem 1 essentially says that even a milder estimate of this kind does not hold for a whole class of continuous exponential sums (i.e. for certain Fourier transforms).

In addition to the usual notation of the theory of distributions (cf. [2], [3], [7]), the following symbols will be used throughout this note. Given a distribution $\Omega \in \mathcal{E}' = \mathcal{E}'(R^n)$, the symbol $[\Omega]$ ($\{\Omega\}$ resp.) denotes the convex hull of the support of $\Omega$ (singular support of $\Omega$, resp.). For $A \subset R^n$, $h_A$ is the supporting function of $A$, i.e. $h_A(\lambda) = \sup_{x \in A} \langle x, \lambda \rangle$, $\lambda \in R^n$. For $\zeta \in C^n$ and $r > 0$, $\Delta(\zeta; r)$ is the closed polydisk with center $\zeta$ and radius $r$; and, if $g(\zeta')$ is any continuous function on $\Delta(\zeta; r)$, we shall write

$$\|g(\zeta)|_r = \max_{\zeta \in \Delta} |g(\zeta)|.$$

2. Indicators of smooth convex bodies.

DEFINITION. Let $\Omega \in \mathcal{E}'$ be such that

$$\{\Omega * T\} = \{0\} + \{T\} \quad (W \in \mathcal{E}).$$

Then $\Omega$ will be called a good convolutor.

The relationship of being a good convolutor to the solvability of the convolution equation $\Phi * u = f$ in the appropriate distribution spaces was discovered by L. Hörmander [7], and since then it was discussed by several authors (for references, cf. [2, Chapter I]). However, it is usually not easy to decide whether a given distribution $\Phi$ is a good convolutor or not.


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$^2$ And more generally, for all exponential polynomials.
Moreover, few good convolutors are known, and as Theorem 1 below will indicate, even distributions of a very simple nature may fail to be good convolutors.

It can be shown [4, Proposition 2] that the following condition on \( \Phi \) is sufficient for \( \Phi \) to be a good convolutor:

**CONDITION** (\( R_\omega \)). There exist constants \( t \geq 0, r > 0, c > 0 \) and \( A \) real (all depending on \( \Phi \)) so that (cf. (1))

\[
|\hat{\Phi}(\zeta)| \geq c(1 + |\zeta|)^r \exp(h_{\Phi}(\eta))
\]

for all \( \zeta = \xi + i\eta \in C^n \) such that \( |\xi| \geq t \) and \( |\eta| > t \log(1 + |\xi|) \).

Since any distribution \( \Phi \) with finite support satisfies condition \( (R_\omega) \) (cf. [4, Proposition 6]), we thus obtain a result of Hörmander [7], [8], according to which all distributions with finite support are good convolutors. This in turn can be used to prove the following statement (cf. [4, Proposition 6]):

Let \( P \) be an arbitrary compact convex polyhedron in \( R^n \) and \( \chi_P \) the distribution defined by the characteristic function of \( P \). Then \( \chi_P \) satisfies condition \( (R_\omega) \), hence \( \chi_P \) is a good convolutor. The same conclusion holds for the surface measure \( \chi_{dP} \) of density 1, i.e.

\[
\chi_{dP}(\phi) = \int_{\partial P} \phi(x) \, ds_x, \quad (\phi \in \mathcal{E})
\]

where \( ds_x \) is the surface element.

It seems natural to ask whether this proposition holds for smooth convex bodies \( P \) as well. At the first glance it seems that it does. Indeed, if, for instance, \( P \) is any ellipsoid in \( R^n \), then the distribution \( \Phi = \chi_P \) satisfies the following weaker version of (2) (cf. the concluding remark in [5]):

\[
(2^*) \{Y\} \subseteq \{\Phi \ast Y\} - \{\Phi\} \quad (\forall Y \in \mathcal{E}).
\]

Therefore, it is rather surprising that this particular \( \Phi \) is not a good convolutor [5, Proposition 4]. The following theorem sheds more light on this peculiar situation.

**THEOREM 1.** Let \( P \) be a convex body in \( R^n (n > 1) \) with a \( C^\infty \)-boundary \( \partial P \). Moreover, it is assumed that the Gaussian curvature of \( \partial P \) never vanishes, i.e. \( K(x) > 0 \) for every \( x \in \partial P \). Then neither \( \chi_P \) nor \( \chi_{dP} \) is a good convolutor.

**REMARK.** Both assumptions on \( \partial P \) (i.e. smoothness and \( K > 0 \)) can be substantially relaxed.

The proof of Theorem 1 is based on a detailed study of the asymptotic behavior of the functions \( \hat{\chi}_P \) and \( \hat{\chi}_{dP} \) in the complex domain. For \( \zeta \) real, estimates of this kind were previously derived by numerous authors (cf. [9], [10], [11] and the references given in [10], [11]). However, for our
purposes these estimates must be sharpened. As an illustration, consider the case of the convex surface \( S = \partial P \). Given \( \zeta = \xi + i \eta \in \mathbb{C}^n \) with \( \xi \neq 0 \), write \( r = |\zeta| \) and consider \( \zeta = r \omega + i \eta \) with \( \omega \) fixed. Let \( x^j = (x^j_1, \cdots, x^j_n) \in S \) \((j=0, 1)\) be the points

\[
x_v^j = \partial h_{S^j}((-1)^j \xi) / \partial \xi_v \quad (v = 1, \cdots, n).
\]

Fix arbitrarily the open subsets \( S^k \) \((k=0, 1, 2)\) of \( S \) so that \( S = \bigcup_k S^k \), \( S^0 \cap S^1 = \emptyset \), \( x^j \in S^j \setminus S^0 \) \((j=0, 1)\). Then for any \( q > n/2 \) and \( v > 0 \) there exist positive numbers \( a_j, b_j \) and \( c_v \) such that

\[
\hat{\chi}_S(\xi) = (1 - i)^{n-1} \left( \frac{\pi}{2} \right)^{n-1} r^{(1-n)/2} \sum K(x^j)^{-1/2} \exp(-i\langle x^j, \xi \rangle)
\]

\[
+ I_1 + I_2 + I_3;
\]

\[
|I_1(\xi)| \leq r^{-n/q}(1 + |\eta|)^q \sum a_j \exp(\langle x^j, \eta \rangle),
\]

\[
|I_2(\xi)| \leq r^{-q}(1 + |\eta|)^{2q} \sum b_j \exp[h_S(\eta)],
\]

\[
|I_3(\xi)| \leq c_v r^{-v}(1 + |\eta|) \exp[h_S(\eta)],
\]

where \( \sum = \sum_{j=0,1} \). Formula (4) combined with a result of Hörmander [8] yields Theorem 1 for \( \chi_{SP} \). Asymptotic expansions similar to (4) hold for \( \hat{\chi}_P \) as well as for the Fourier transforms of certain measures with non-constant density.

3. The discrete case. Generalization of Ritt's theorem. In this part we shall consider finite exponential sums, and more generally, exponential polynomials in several complex variables. If \( H \) is an exponential polynomial, i.e. a function of the form

\[
H(\xi) = \sum_{j=1}^g h_j(\xi) \exp(\langle \theta_j, \xi \rangle) \quad (\xi \in \mathbb{C}^n)
\]

with complex frequencies \( \theta_j \in \mathbb{C}^n \) and polynomial coefficients \( h_j \), the greatest common divisor of the \( h_j \)'s, \( d_H = (h_1, \cdots, h_g) \), will be called the content of \( H \). Moreover, we shall write \( C_H(\xi) = \max \Re(\theta_j, \xi) \). Henceforth an exponential sum will mean a function of the form (5) with all coefficients \( h_j \) constant. The following lower estimate of exponential polynomials was proved in [3], [5]:

\( (R_0) \) Given an exponential polynomial \( H \) and an arbitrary \( \varepsilon > 0 \), there exists \( C = C(\varepsilon, H) > 0 \) such that for every \( \xi \in \mathbb{C}^n \) and any \( f \) analytic in \( \Delta(\xi; \varepsilon) \),

\[
|f(\xi)| \exp(C_H(\xi)) \leq C |f(\xi) H(\xi)| \varepsilon.
\]

\( ^* \) Obviously, estimate \( (R_0) \) is much stronger than \( (R_\omega) \).
In this section we shall discuss the following

**Question.** Let $F$ and $G$ be exponential polynomials in $n$ variables such that the function $H = F/G$ is entire. What can be said about the structure of $H$? In particular, when is $H$ an exponential polynomial?

Simple examples show that $H$ need not be an exponential polynomial (e.g., $n=1, F = \sin \zeta, G = \zeta$). On the other hand, if $F$ and $G$ are exponential sums in one variable such that $H$ is entire, then, according to a theorem of Ritt [12], $H$ is also an exponential sum. Different proofs of Ritt's theorem were given by H. Selberg, P. D. Lax and A. Shields (cf. the references in [12], [13]). In particular, Shields [13] proves that $H$ is an exponential polynomial as long as it is entire and $G$ is an exponential sum. He also mentions that, according to an unpublished result of W. D. Bowsma, the last assumption may be replaced by $d_G = 1$. Finally, Avanissian and Martineau [1] generalized the original Ritt's theorem to arbitrary $n > 1$.

The following theorem contains all these results as special cases. Moreover, it shows that the above counterexample is in a certain sense the best possible:

**Theorem 2.** Let $F, G, H$ be as above ($n \geq 1$ arbitrary). Then there exists an exponential polynomial $E$ and a polynomial $Q$ such that $H = E/Q$. Hence we may assume $(d_E, Q) = 1$. Then $E$ and $Q$ are determined uniquely* and $Q$ divides $d_G$.

The starting point for the proof of Theorem 2 is the following assertion: Let $f, g, h$ be the analytic functionals whose Fourier-Borel transforms are $F, G, H$ respectively. Then $h$ is carried by the polyhedron defined by $\mathbb{C}_F - \mathbb{C}_G$. This in turn follows from (R0).

The proofs together with applications of the above theorems will appear elsewhere.

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* Up to a constant multiple.


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