AN ILL-POSED PROBLEM FOR A STRICTLY HYPERBOLIC EQUATION IN TWO UNKNOWNS NEAR A CORNER

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In an earlier note [4] we gave a simple example of an ill-posed problem for a system of hyperbolic equations in a region whose boundary has a corner. The system was diagonal with coupling only at the boundary. Earlier we derived necessary and sufficient conditions for well-posedness [2] for a wide class of constant coefficient hyperbolic systems in such regions. In [3] we examined in some detail the phenomena which occur when these conditions are violated. The fundamental work for hyperbolic problems in regions with smooth boundaries was done by Kreiss [1].

It was pointed out by Sarason and Smoller [5] that the work of Strang [6] for the half-space problem implies that the corner problem is well posed for a strictly hyperbolic system in two unknowns iff the corresponding half-space problems are well posed. They constructed, using geometrical optics, a four dependent variable ill-posed example, where the half-space extensions were well posed.

In all the above-mentioned work, the boundary conditions imposed were local, i.e., of the form \( Bu - f \) at \( x_1 = 0 \), where \( B \) is a matrix and \( u \) is the unknown vector on the boundary.

We have noticed that much of the theory can be extended to nonlocal pseudo-differential boundary conditions. In particular, conditions of the form

\[
B(w_2, \cdots, w_n, s)\hat{u}(0, w_2, \cdots, w_n, s) = f(w_2, \cdots, w_n, s),
\]

where \( B \) is a matrix-valued function of the dual variables \( x_1 \rightarrow w_1, t \rightarrow s \), can be treated. Such boundary conditions are reasonable when nonlinear problems are linearized. We shall discuss this in detail in a future paper.

Our purpose here is to show that for such boundary conditions well-posedness of the two half-space problems does not imply well-posedness of the corner problem, even in the strictly hyperbolic two unknown variable case.

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We shall present necessary and sufficient algebraic conditions for well-posedness of this problem in the above-mentioned paper.

We consider the equation

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  u \\
  v
\end{pmatrix} + \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  u \\
  v
\end{pmatrix} + \begin{pmatrix}
  F_1 \\
  F_2
\end{pmatrix}
\]

to be solved for the complex-valued functions \( u \) and \( v \) in the region \( 0 < x, y, t \) with initial conditions

\[
u(x, y, 0) = v(x, y, 0) = 0.
\]

Next, apply a Laplace transform in \( t \), use the initial conditions (2), and call the dual variable \( s = \eta + i \xi \), with \( n > 0, \xi \) real. We have

\[
s \hat{U} - A \hat{U}_x - B \hat{U}_y = \hat{F},
\]

where \( \hat{U} \) and \( \hat{F} \) are the transformed 2 vectors \( \left( \frac{\hat{u}}{\hat{v}} \right) \) and \( \left( \frac{\hat{F}_1}{\hat{F}_2} \right) \), respectively; \( A \) and \( B \) are defined in (1).

We impose boundary conditions for \( \eta > 1 \):

\[
\begin{align*}
(a) \quad & \hat{u}(0, y, s) = -\frac{1 + \sqrt{(1 - c^2)}}{c} \Phi_1(\xi) \hat{v}(0, y, s) + f(y, s), \\
(b) \quad & \hat{u}(x, 0, s) = -\frac{1 + \sqrt{(1 - c^2)}}{c} \Phi_2(\xi) \hat{v}(x, 0, s) + \hat{g}(x, s),
\end{align*}
\]

where \( c \) is any real number, \( 0 < c < 1 \), \( \Phi_1(\xi) \), \( \Phi_2(\xi) \) are \( C_0^\infty(-\infty, \infty) \) with \( 0 \leq \Phi_1 \leq 1 \), \( -1 \leq \Phi_2 \leq 1 \), and \( \Phi_1(\xi) \equiv 1 \) if \( -\frac{1}{2} < \xi < \frac{1}{2} \), \( \Phi_1(\xi) \equiv 0 \), \( \Phi_2(\xi) \equiv -1 \) if \( |\xi| > 1 \).

The standard estimate for problems of this type is

\[
(\eta - \eta_0) \| \hat{U} \|^2 + \| \hat{U} \|^2_B \leq K(\| \hat{F} \|^2 + \| f \|^2_B + \| g \|^2_B)
\]

uniformly in \( s = \eta + i \xi \), for \( \eta > \eta_0, \eta_0 > 0 \) and fixed.

The norms are defined as

\[
\| \hat{U} \|^2 = \int_0^\infty \int_0^\infty (|\hat{u}(x, y, s)|^2 + |\hat{v}(x, y, s)|^2) \, dx \, dy,
\]

\[
\| \hat{U} \|^2_B = \int_0^\infty [ |\hat{u}(0, y, s)|^2 + |\hat{v}(0, y, s)|^2 + |\hat{u}(y, 0, s)|^2 + |\hat{v}(y, 0, s)|^2 ] \, dy.
\]

\( \| \|_B \) and \( \| \|_B^2 \) are defined analogously.

We have the following

**Theorem.** No estimate of type (5) is possible for problem (3), (4)(a), (b). However, problem (3), (4)(a) in the region \( 0 < x, -\infty < y < \infty, \) and
(3), (4)(b), in the region $-\infty < x < \infty$, $0 < y$ both obey estimates of type (5), where the norms are modified in an obvious fashion, to be integrals over half- rather than quarter-space.

PROOF. For the quarter-space problem, we consider

$$\hat{U} = \exp(-csy - sx\sqrt{1 - c^2}) \left[ -\frac{1}{c/(1 + \sqrt{1 - c^2})} \right]$$

for $s = \eta + i\xi$, $0 < \eta$, $-\frac{1}{2} < \xi < \frac{1}{2}$. This function satisfies the homogeneous equations (3), (4)(a), (b). Moreover, the norms on the left side of (5) are finite.

For the right half-plane problem we can easily obtain the estimate

$$\|\hat{U}\|^2 \leq K_1(\|\hat{F}\|^2 + \|\hat{U}\|_B^2)$$

independently of the boundary conditions (4)(a). (See e.g., Osher [2].)

We need only to obtain

$$\|\hat{U}\|_B^2 \leq K_2(\|\hat{F}\|^2 + \|\hat{F}\|_B^1 + \|\hat{U}\|^2).$$

Moreover, in a standard fashion, we can assume $\hat{F} = 0$. (See, e.g., Osher [2].)

We can solve equation (3) for $F = 0$, with boundary conditions (4)(a). Fourier transform (3) in $y$, then multiply by $A^{-1}$. We have

$$\check{U}_x = A^{-1}(s - Biw)\check{U} = 0,$$

where $w$ is the dual variable, $\check{U} = \mathcal{F} \hat{U}$,

$$(b) \quad \check{u}(0, w, s) = -\frac{1}{c} + \frac{\sqrt{1 - c^2}}{c}\Phi_1(\xi)\check{b}(0, w, s) + \check{f}(w, s).$$

Let

$$\check{U} = T_1(w, s)\check{F}.$$  

$T_1(w, s)$ is a unitary matrix-valued measurable function of $w, s$ such that

$$T_1^*(A^{-1}(s - Biw))T_1 = \begin{pmatrix} -K_+ & m_{12}(w, s) \\ 0 & K_+ \end{pmatrix}$$

where $K_+ = \sqrt{(s^2 + w^2)}$, Re $K_+ > 0$.

Thus, the general solution to (10)(a) which does not grow exponentially as $x \to +\infty$, is

$$\check{F} = \begin{pmatrix} \exp(-K_+ x) \\ 0 \end{pmatrix} b_1(w, s),$$
or

\begin{equation}
\mathcal{U}(0, w, s) = T_1(w, s) \begin{pmatrix} b_1(w, s) \\ 0 \end{pmatrix}.
\end{equation}

Apply the boundary condition (10)(b). Thus

\begin{equation}
b_1(w, s) = \frac{\hat{f}(w, s)[|s + K_+|^2 + |w|^2]^{1/2}}{\left[ (s + K_+) + \frac{1 + \sqrt{(1 - c^2)}}{c} \cdot iw\Phi_1 \right]}.
\end{equation}

It is easy to show that the quantity multiplying \(\hat{f}(w, s)\) is uniformly bounded in \(w, s\), if \(\eta > 1\). Thus, this half-space problem is well posed.

We can do the analogous thing for the upper half-plane problem, arriving at

\begin{equation}
\mathcal{U}(w, 0, s) = T_2(w, s) \begin{pmatrix} b_2(w, s) \\ 0 \end{pmatrix}.
\end{equation}

Applying the boundary condition at \(y = 0\) leads to

\begin{equation}
b_2(w, s) = \frac{\tilde{g}(w, s)(|s - iw| + |s + iw|)^{1/2}}{\left[ \sqrt{s - iw} - \frac{1 + \sqrt{(1 - c^2)}}{c} \cdot \sqrt{s + iw}\Phi_2(\xi) \right]},
\end{equation}

where each square root has positive real part. Again we have \(|b_2(w, s)| \leq K_2|\tilde{g}(w, s)|\) if \(\eta > 1\). Thus the half-space problem is well posed. Q.E.D.

**BIBLIOGRAPHY**


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