A function \( f(z) \) analytic in the unit disk is said to belong to the Bergman space \( A^p \) (\( 0 < p < \infty \)) if \( \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r \, dr \, d\theta < \infty \). It is clear that \( A^p \) contains the Hardy space \( H^p \) of analytic functions for which \( \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty \). We adopt the convention that \( A^\infty = H^\infty \), the space of bounded analytic functions in the disc.

Assuming that \( f(0) \neq 0 \), we list the zeros of \( f \) in order of nondecreasing modulus: \( 0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots < 1 \). We repeat \( z_i \) according to the multiplicity of the zero of \( f \) at \( z_i \). The sequence \( \{z_i\} \) is called the zero set of \( f \). If \( f \in A^p \) (resp. \( H^p \)), then \( z_i \) will be called an \( A^p \) (resp. \( H^p \)) zero set. It has long been known that \( H^p \) zero sets (\( 0 < p \leq \infty \)) are completely characterized by the condition \( \prod_{k=1}^\infty 1/|z_k| < \infty \). (Equivalently, \( \sum_{k=1}^\infty 1 - |z_k| < \infty \).) In particular, the condition is independent of \( p \).

Our results show that the situation for \( A^p \) zero sets is considerably more complex.

**LEMMA 1.** If \( \{z_k\} \) is an \( A^p \) zero set (\( 0 < p < \infty \)), then
\[
\frac{1}{N} \prod_{k=1}^N \frac{1}{|z_k|} = O(N^{1/p}).
\]

**COROLLARY.** If \( \{z_k\} \) is an \( A^p \) zero set (\( 0 < p < \infty \)), then for each \( \varepsilon > 0 \),
\[
\sum_{k=1}^\infty (1 - |z_k|) \left( \log \frac{1}{1 - |z_k|} \right)^{-1-\varepsilon} < \infty.
\]

If \( f(z) = \sum_{n=0}^\infty a_n z^n \), let \( S_N^{(p)} = \sum_{k=1}^N |a_k|^p \), \( p > 0 \).

**LEMMA 2.** If \( S_N^{(p)} = O(N^a) \) for some \( a \geq 1 \), then \( f \in A^p \) for all \( p < 2/a \).

**LEMMA 3.** For some \( p \), \( 1 \leq p \leq 2 \), suppose that \( \sum_{n=1}^\infty n^{-p}S_n^{(p)} < \infty \) and \( N^{1-p}S_n^{(p)} = O(1) \). Then \( f \in A^{p'} \), \( 1/p + 1/p' = 1 \).

Lemma 1 is proved by an application of Jensen’s theorem. Lemmas 2 and 3 follow from corresponding coefficient conditions, after a summation by parts. In particular, Lemma 3 is a consequence of the fact that...
\[ \sum |a_N|N < \infty \text{ implies } f \in A^\infty, \text{ and that } \sum |a_N|^2N < \infty \text{ implies } f \in A^2. \]

One merely applies the Riesz interpolation theorem and summation by parts to obtain the result.

**Theorem 1.** Let \( 0 < p < q \leq \infty \). Then there exists an \( A^p \) zero set which is not an \( A^q \) zero set.

**Sketch of Proof.** Let \( f(z) = \prod_{k=0}^{\infty} 1 + uz^k \), where \( b \) is an integer greater than 2, and \( u \) is a positive constant. Using the notation of the lemmas, one verifies that:

1. Every partial product for \( f(z) \) is a partial sum of its Taylor series.
2. If \( N = \sum_{k=1}^{b^k} b^k, S_N^p = (1 + u^p)^s \).
3. If \( u > 1 \), if \( N = \sum_{k=1}^{b^k} b^k \), and if \( \{z_i\} \) are the ordered zeros of \( f \), then \( \prod_{i=1}^{N} 1/|z_i| = u^s \).

From these facts, and from Lemmas 1, 2 and 3, we conclude that:

4. If \( b \leq 1 + u^2 \), then \( f \in A^p \) for all \( p < 2 \log b/\log(1 + u^2) \). (Also, in this case, \( f \notin A^2 \)).
5. If \( 1 + u^s \leq b^{s'-1} \) for some \( s, 1 < s \leq 2, f \in A^p \) for all \( p < s' \), where \( 1/s + 1/s' = 1 \).
6. If \( u > 1 \), the zero set of \( f \) is not the zero set of any function in \( A^q \) for \( q > \log b/\log u \).

An examination of (4), (5) and (6) shows that if \( 0 < p < q \leq \infty \), \( u \) and \( b \) may always be chosen to yield a function \( f \) in \( A^p \) whose zero set is not an \( A^q \) zero set.

**Theorem 2.** For \( 0 < p < \infty \), the union of two \( A^p \) zero sets is not in general an \( A^p \) zero set.

To prove Theorem 2, we choose one of the functions \( f \in A^p \) constructed in Theorem 1, with the parameter \( u > 1 \). We choose a positive integer \( N \) and require that each zero of \( f \) be repeated \( N \) times. For \( N \) sufficiently large we obtain a sequence which, by Lemma 1, cannot be an \( A^p \) zero set.

We state two corollaries to the above theorems, both of which again contrast sharply with \( H^p \) theory.

**Corollary (to Theorem 1).** It is not possible to represent an arbitrary \( A^1 \) function as the product of two functions in \( A^2 \), one of them nonvanishing.

**Corollary (to Theorem 2).** Consider the operator \( M_z \) of multiplication by \( z \) on \( A^6 \) (a weighted unilateral shift). There exist two nontrivial closed invariant subspaces of \( M_z \) whose intersection is trivial.