SETS OF COLORINGS OF CIRCUITS

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1. Introduction. A circuit $\Gamma$ is a triangulation of the one-dimensional sphere $S^1$. It shall have as its set of vertices $\Gamma _0 = \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$, and as its set of one-simplices $\Gamma _1 = \{\sigma _j = (j-1, j) | j = 1, 2, \ldots, k\}$. A coloring of $\Gamma$ is a zero-dimensional cochain $c^0 \in C^0(\Gamma, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ whose coboundary is "nowhere zero", i.e. $\delta c^0(\sigma _j) \neq 0$ for all $\sigma _j \in \Gamma _1$. A set $K$ of colorings of $\Gamma$ is realizable as a set of admissible colorings if there is a triangulated two-dimensional disk $D$ with boundary $\Gamma$ such that the restriction homomorphism

$$j^\#: C^0(D, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \rightarrow C^0(\Gamma, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$$

(induced by the inclusion $j: \Gamma \rightarrow D$) takes the colorings of $D$ onto $K$.

Let $\psi(k)$ be the minimum cardinality of a set $K$ which is realizable as a set of admissible colorings.

Remark 1. $\psi(k)=0$ if and only if the four color conjecture is false.

The conjecture of Albertson and Wilf [1]. $\psi(k)=3 \cdot 2^k$ for $k=3, 4, \ldots$.

Comment 1. Since $3 \cdot 2^k$ is the number of colorings of any disk $D$ with no interior vertices and $k$ vertices in $\Gamma_0 = \mathbb{Z}_k$, we conclude $3 \cdot 2^k \geq \psi(k)$.

Comment 2. It is not known whether the four color conjecture implies the Albertson-Wilf conjecture for $k>6$. (It does for $k=3, 4, 5$ and 6 [1].)

In [1], Albertson and Wilf announce:

Theorem 1. If the four color conjecture holds then

$$\psi(k) \geq (4!)^{F_{k-1}} \geq (1 + \sqrt{5})/2)^k$$

where $F_k$ is the $k$th Fibonacci number.

By generalizing the notion of a set of admissible colorings of $\Gamma$ to the notion of a complete set of colorings of $\Gamma$, one can prove by induction on $k$:

Theorem 2. If the four color conjecture holds then

$$\psi(k) > 4 \cdot 3^{k/2} \quad \text{if } k \text{ is even},$$

$$> 8 \cdot 3^{(k-1)/2} \quad \text{if } k \text{ is odd}.$$
2. Coboundaries of colorings. Let $D$ be a triangulated two-dimensional disk with boundary $\Gamma$. Since $D$ is connected, $H^0(D, Z_2 \oplus Z_2) \approx Z_2 \oplus Z_2$. Hence there are exactly four colorings corresponding to each nowhere zero one-dimensional cobounding cocycle. All of the sets of colorings that we consider will contain all four colorings with a given coboundary if the set contains any one of them. Thus we can consider the sets of coboundaries of colorings as easily as the sets of colorings.

The disk $D$ is contractible so $H^1(D, Z_2 \oplus Z_2) \approx 0$. Hence the group of cocycles $Z^1(D, Z_2 \oplus Z_2)$ is equal to the group of cobounding cocycles $B^1(D, Z_2 \oplus Z_2)$. [In this case the cobounding cocycles are characterized by the sum of all values being zero.]

Notation. $Z_2 \oplus Z_2 = \{0, e_1, e_2, e_3\}$ with the obvious addition.

If $z$ is a nowhere zero cocycle on $D$ and $\tau^2$ is a two-simplex with faces $\alpha, \beta$ and $\gamma$, then $z(\alpha) + z(\beta) + z(\gamma) = 0$. Hence $z$ assigns the three values $e_1, e_2$ and $e_3$ to the faces $\alpha, \beta$ and $\gamma$ of $\tau^2$. Let us suppose that $z(\beta) = e_3$. We may hold the value $e_3$ fixed and interchange the values $e_1$ and $e_2$ on $\alpha$ and $\gamma$. This change will propagate along a $Z_2$ cocycle which contains either no one-simplexes from $\Gamma_1$ or exactly two one-simplexes from $\Gamma_1$.

REMARK 2. Let $z$ be a nowhere zero cocycle on $D$ and $e_i$ a fixed value in $Z_2 \oplus Z_2$. For each $\sigma_j$ in $\Gamma_1$ with $z(\sigma_j) \neq e_i$, there is a uniquely determined $\sigma_j' \in \Gamma_1$ and $z'$ a nowhere zero cocycle on $D$ such that:

(i) $j' \neq j$.
(ii) For every $\alpha \in D_1$, $z'(\alpha) = e_i$ if and only if $z(\alpha) = e_i$.
(iii) For every $\sigma_j \in \Gamma_1$

$$z'(\sigma_j) = z(\sigma_j) + e_i \quad \text{if } l = j, j',$$

$$z(\sigma_j) \quad \text{otherwise.}$$

The pairing $\sigma_j \leftrightarrow \sigma_j'$ is called a planar change diagram for $\tau^2(z)$ and $e_i$. [$j^\#(z) \in Z^1(\Gamma, Z_2 \oplus Z_2).$] It satisfies:

(i) If $\sigma_j \leftrightarrow \sigma_j'$, then neither $z(\sigma_j) = e_i$ nor $z(\sigma_j') = e_i$. Furthermore if $z(\sigma_j) \neq e_i$ then $\sigma_j$ belongs to a pair.
(ii) If $\sigma_j \leftrightarrow \sigma_j'$ and $\sigma_i \leftrightarrow \sigma_i'$ then $\sigma_i$ and $\sigma_i'$ lie on the same arc between $\sigma_j$ and $\sigma_j'$.

Let $z$ be a nowhere zero cobounding cocycle on $\Gamma$ and let $P$ be a planar change diagram for $z$ and $e_i$. With each set of pairs of $P$ we can associate a nowhere zero cobounding cocycle $z'$ on $\Gamma$. This association is called the action of $P$ on $z$. If $z$ is in some set $L$ of cocycles and $z' \in L$ for all sets of pairs of $P$ then we say $L$ is closed under the action of $P$.

DEFINITION. A complete set $K$ of colorings of $\Gamma$ corresponds to a set $\delta K$ of nowhere zero cobounding cocycles with the properties:

(i) $K$ is invariant under the action of the six automorphisms $\nu: Z_2 \oplus Z_2 \mapsto Z_2 \oplus Z_2$. 


(ii) For each \( z \in \delta K \) and \( e_i \), there is a planar change diagram \( P \) so that \( \delta K \) is closed under the action of \( P \).

3. **Induced sets.** A nondegenerate simplicial map \( f: E \to F \) induces a homomorphism: \( f^\#: B^1(F, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \to B^1(E, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \), which preserves the property of being nowhere zero. In general, however, complete sets of colorings on circuits are not preserved. [Let \( f: \Gamma' \to \Gamma \) be a two-fold covering.]

4. **Potted trees.** A potted tree is a contractible simplicial complex with no more than one two-dimensional simplex. A circuit \( \Gamma \) is properly mapped to a potted tree \( T \) if \( f: \Gamma \to T \) is a nondegenerate simplicial map such that for each \( \alpha \in T_1 \), \( f^{-1}(\alpha) \) has exactly one or exactly two elements depending upon whether \( \alpha \) is the face of a two simplex or not.

**Remark 3.** If \( \Gamma \) is properly mapped to the potted tree \( T \) then \( T \) has \( k/2 \) or \( (k+3)/2 \) elements. The set \( L \) of all colorings of \( T \) has \( n(k) \) elements and \( f^\#(L) \) is a complete set of colorings of \( \Gamma \) of cardinality \( n(k) \) where

\[
  n(k) = \begin{cases} 
    4 \cdot 3^{k/2} & \text{if } k \text{ is even}, \\
    8 \cdot 3^{(k-1)/2} & \text{if } k \text{ is odd}.
  \end{cases}
\]

**Comment 3.** \( n(k) \leq 2 \cdot n(k-1) \) [equality when \( k \) is odd]; \( n(k) = 3 \cdot n(k-2) \).

A set \( K \) of colorings of \( \Gamma \) is realizable as induced by a potted tree if there exists a proper map \( f: \Gamma \to T \) such that \( K = f^\#(L) \).

5. **Outline of the proof of Theorem 2.** From a circuit \( \Gamma \), we can form a circuit \( \Gamma' \) by deleting the open star of the vertex \( 1 \) and by inserting a one-simplex \((0, 2)\). We can also form a circuit \( \Gamma'' \) by performing the same deletion and identifying the vertices \( 0 \) and \( 2 \). Since every nowhere zero cobounding cocycle \( z \) on \( \Gamma \) induces either a cocycle \( z' \) on \( \Gamma' \) or a cocycle \( z'' \) on \( \Gamma'' \), a complete set of colorings \( K \) on \( \Gamma \) induces complete sets \( K' \) and \( K'' \) on \( \Gamma' \) and \( \Gamma'' \), respectively.

We prove by induction on the number \( k \) of vertices of \( \Gamma \), that \( K \) has fewer than \( n(k) \) elements only if \( K \) is empty. This follows from two inequalities. First the number of elements in \( K'' \) is less than or equal to one third the number of elements in \( K \). Secondly, if \( K'' \) is empty then the number of elements in \( K' \) is less than or equal to half the number of elements in \( K \). In essentially the same way we can prove that if \( K \) has exactly \( n(k) \) elements then \( K \) can be realized as induced by a potted tree.

**Reference**


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