A REMARK CONCERNING PERFECT SPLINES

BY CARL DE BOOR

Communicated by R. C. Buck, January 3, 1974

Let \( x := (x_i) \) be nondecreasing. For a sufficiently smooth \( f \), denote by \( f|_x := (f_i) \) the corresponding sequence given by the rule

\[
f_i := f^{(j)}(x_i) \quad \text{with} \quad j = j(i) := \max \{ m \mid x_{i-m} = x_i \}.
\]

Assuming that \( x \) is in \([a, b]\) and that \( x_i < x_{i+n} \), all \( i, f|_x \) is defined for every \( f \) in the Sobolev space

\[
W^{(n)}_\infty[a, b] := \{ f \in C^{(n-1)}[a, b] \mid f^{(n-1)} \text{ abs. cont.}; f^{(n)} \in L_\infty[a, b] \}.
\]

Karlin [6] discusses the problem of minimizing \( \| f^{(n)} \|_\infty \) over all \( f \) in \( \Pi(x, \alpha) := \{ f \in W^{(n)}_\infty \mid f|_x = \alpha \} \) for a given sequence \( \alpha \), and announces the following

**THEOREM (S. KARLIN [6]).** Let \( x = (x_i)^{n+r} \) be a given nondecreasing sequence in the finite interval \([a, b]\), with \( x_i < x_{i+n} \), all \( i \). Let \( \alpha \in R^{n+r} \) be given. Then \( \Pi(x, \alpha) \) contains a perfect spline of order \( n \) with less than \( r \) (interior) knots, i.e., a function of the form

\[
p(x) = \sum_{i=0}^{n-1} a_i x^i + c \left[ x^n + 2 \sum_{i=1}^{k-1} (-1)^i (x - \xi_i)^n \right]
\]

for some real constants \( a_0, \ldots, a_{n-1}, c, \) and for \( a < \xi_1 < \cdots < \xi_{k-1} < b \) with \( k \leq r \). Further, \( \| f^{(n)} \|_\infty \) takes on its minimum value over \( f \in \Pi(x, \alpha) \) at this \( p \).

It is the purpose of this note to outline a simple proof of this theorem.

For this, denote by \([x_i, \ldots, x_{i+n}] f \) the \( n \)th divided difference of \( f \) at the \( n+1 \) points \( x_i, \ldots, x_{i+n} \). Then \( [x_i, \ldots, x_{i+n}] (f-g) = 0 \) for all \( f, g \in \Pi(x, \alpha) \) and \( i = 1, \ldots, r \). Further, it is well known (see e.g., [2]) that, for \( f \in W^{(n)}_4[a, b] \),

\[
[x_i, \ldots, x_{i+n}] f = \int_a^b \varphi_i(t) f^{(n)}(t) \, dt
\]

with

\[
\varphi_i(t) := M_i, n(t)/n! := [x_i, \ldots, x_{i+n}] (\cdot - t)^{n-1}/(n - 1)!
\]
a (polynomial) \( B \)-spline of order \( n \) having the knots \( x_i, \ldots, x_{i+n} \). Hence,
PERFECT SPLINES

275

with $f_0$ any particular element of $\Pi(x, \alpha)$, $\Pi(x, \alpha)$ is contained in

$$\left\{ f \in W^{(n)}_\infty \left| \int_a^b \varphi_i(t)(f - f_0)^{(n)}(t) \, dt = 0, i = 1, \ldots, r \right. \}.$$  

On the other hand, for every $f$ in the set (2), there exists a polynomial $p_f$ of degree $< n$ so that $f - p_f \in \Pi(x, \alpha)$, viz. the unique polynomial $p_f$ of degree $< n$ for which

$$p_f|_{(a,b)^*} = (f - f_0)|_{(a,b)^*}.$$  

Consequently, with the definition

$$\Pi^{(n)}(x, \alpha) := \left\{ g \in L_\infty[a,b] \left| \int \varphi_i g = \int \varphi_i f_0^{(n)}, \; i = 1, \ldots, r \right. \},$$

it follows that

$$\inf \{ \| f^{(n)} \|_\infty \left| f \in \Pi(x, \alpha) \right. \} = \inf \{ \| g \|_\infty \left| g \in \Pi^{(n)}(x, \alpha) \right. \},$$

and that $n$-fold differentiation maps the set of solutions of the left-hand minimization problem one-one and onto the set of solutions of the right-hand minimization problem. Equation (3) can already be found in Favard's pioneering paper [3].

It remains to show that $\Pi^{(n)}(x, \alpha)$ contains a function of constant absolute value and with less than $r$ sign changes, and which solves the right-hand minimization problem in (3). For this, we use the idea, apparently due to M. G. Krein [7], of looking at constrained minimization dually, as a problem of finding norm preserving extensions for a given linear functional, and then using representation theorems for such functionals. Consider the linear functional $\lambda_0$ defined on

$$S_{n,x} := \text{span}(\varphi_1, \ldots, \varphi_r) \subseteq L_1[a,b]$$

by the rule

$$\lambda_0 \varphi := \int_a^b \varphi(t)f_0^{(n)}(t) \, dt, \; \; \; \text{all } \varphi \in S_{n,x}.$$  

Then, identifying $L_\infty[a,b]$ with the continuous dual of $L_1[a,b]$ in the usual way, $\Pi^{(n)}(x, \alpha)$ is seen to coincide with the collection of all extensions of $\lambda_0$ to a continuous linear functional on $L_1[a,b]$. Hence

$$\inf \{ \| g \|_\infty \left| g \in \Pi^{(n)}(x, \alpha) \right. \} = \inf \{ \| \lambda \| \left| \lambda \in (L_1[a,b])^*, \lambda|_{S_{n,x}} = \lambda_0 \} = \| \lambda_0 \|$$

by the Hahn-Banach theorem, settling existence of a solution as well. Let $\psi$ be an element of $S_{n,x}$ of unit 1-norm at which $\lambda_0$ takes on its norm, i.e., let

$$\psi \in S_{n,x}, \; \; \; \| \psi \|_1 = 1, \; \; \; \lambda_0 \psi = \sup_{\varphi \in S_{n,x}} \lambda_0 \varphi / \| \varphi \|_1,$$
and set \( h := (\lambda_0 \psi) \text{signum} \psi \). If \( g \) is any point in \( \Pi^{(n)}(x, \alpha) \) at which the infimum is taken on, then

\[
\|g\|_\infty = \|\lambda_0\| = \lambda_0 \psi = \int \psi g \leq \|\psi\|_1 \|g\|_\infty = \|g\|_\infty
\]

and equality must therefore hold in Hölder’s inequality. This implies that

\[ g(t) = \|g\|_\infty \text{signum} \psi(t) = h(t) \quad \text{a.e. on} \{t \mid \psi(t) \neq 0\}. \]

Hence, if \( \psi \) vanishes only on a set of measure zero, then \( h \) is an element of \( \Pi^{(n)}(x, \alpha) \) of constant absolute value and is the unique element of \( \Pi^{(n)}(x, \alpha) \) at which \( \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(x, \alpha)\} \) is attained.

This simple idea is at the bottom of Glaeser's successful treatment [4] of the special case

\[ r = n, \quad x_1 = \cdots = x_n = a, \quad x_{n+1} = \cdots = x_{2n} = b. \]

In this special case, \( S_{n,*} \) reduces to the space of polynomials of degree \( <r \), hence every nonzero \( \psi \in S_{n,*} \) vanishes only at \( <r \) points.

In the general case, \( \psi \in S_{n,*} \) is known to vanish only at \( <r \) points unless \( \psi \) vanishes on an interval. But whether or not this happens, with \( S_e := K_e(S_{n,*}) \subseteq L_1[a, b] \), where \( e > 0 \) and

\[
(K_e g)(x) := \int_{-\infty}^{\infty} \exp(-\xi^2/(2e^2)) g(\xi) d\xi / (e\sqrt{2\pi}),
\]

every nonzero \( \psi \in S_e \) vanishes only at \( <r \) points [5, proof of Theorem 4.1 in Chapter 10, especially item (4.23)]. Hence, there exists exactly one \( h_\epsilon \) in

\[ \Pi_\epsilon := \left\{ g \in L_\infty[a, b] \mid \left. \int \varphi_\epsilon g = \int \varphi_\epsilon f^{(n)} \right\} \forall \varphi_\epsilon \in S_e \right\}
\]

at which \( \inf\{\|g\|_\infty \mid g \in \Pi_\epsilon\} \) is attained, and this \( h_\epsilon \) is of constant absolute value and has fewer than \( r \) sign changes. Since

\[ \lim_{\epsilon \to 0^+} \|\varphi - K_\epsilon \varphi\|_1 = 0 \quad \text{for all} \ \varphi \in S_{n,*}, \]

it follows that

\[ \liminf_{\epsilon \to 0^+} \|h_\epsilon\|_\infty \leq \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(x, \alpha)\}, \]

hence, for some positive null sequence \( (e_m) \) and some points \( \xi_1, \dotsc, \xi_{k-1} \) in \( [a, b] \) with \( k \leq r \), \( (h_{e_m}) \) converges uniformly on compact subsets of \( [a, b] \) \{\( \xi_1, \dotsc, \xi_{k-1} \) \} to some function \( h \) for which

\[ \lim_{m \to \infty} \|h_{e_m}\|_\infty = \|h\|_\infty \leq \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(x, \alpha)\}, \]
But this $h$ is necessarily of constant absolute value, has fewer than $r$ sign changes and is in $II(n)(x, \alpha)$, which finishes the proof.

The above argument extends at once to the minimization of $\|Lf\|_\infty$ under the same constraints, with $L$ an $n$th order ordinary linear differential operator which is totally disconjugate.

**Acknowledgement.** The happy thought of using some smoothing in the study of best approximation by splines I picked up from Barrar and Loeb [1]. Much of the above became clear to me during a recent visit to the Los Alamos Scientific Laboratory in New Mexico.

**References**