EXTREMAL LENGTH, REPRODUCING DIFFERENTIALS
AND ABEL'S THEOREM

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Let $c$ be a 1-chain on a Riemann surface $R$ and $\Gamma_a(R)$ a closed subspace of $\Gamma^a(R)$, the Hilbert space of square integrable harmonic differential forms on $R$, then there is a unique $\psi_a(c) \in \Gamma^a(R)$ such that $\int_{\partial} \omega = (\omega, \psi_a(c))$ for all $\omega \in \Gamma^a(R)$. $\psi_a(c)$ is called the $\Gamma^a(R)$-reproducing differential for $c$ and $\|\psi_a(c)\|^2$ is a conformal invariant. For the case of a 1-cycle $c$ an extremal length interpretation for the squared norm of the reproducing differential was given by Accola [1] and Blatter [2] for $\Gamma^b(R)$, by Marden [3] for $\Gamma^h(R)$ and by Rodin [5] for $\Gamma^h(R)$. In each of these results the curve family whose extremal length gave the square of the norm of the reproducing differential was a homology class associated with $c$. Rodin [5] asked whether there were similar theorems for other subspaces of $\Gamma^a(R)$ and what the proper curve family would be in case $c$ was an arbitrary 1-chain, not necessarily a 1-cycle. If $c$ is a single arc, then a reduced extremal distance interpretation of the norm of the reproducing differential for $\Gamma^h(R)$, $\Gamma^m(R)$ and $\Gamma^h(R) \cap \Gamma^h(R)$ was given in [4]. The purpose of this paper is to announce solutions to the problems posed by Rodin for a large number of important subspaces of $\Gamma^h(R)$; a complete, detailed paper is forthcoming.

For the sake of simplicity we shall consider only compact Riemann surfaces; this case gives rise to one of the most important applications. Let $c$ be a 1-chain on the compact Riemann surface $R$. Suppose that $\partial c = \sum_{i=1}^{J} n_i b_j - \sum_{i=1}^{J} m_i a_i$, where the points $a_i$, $b_j$ are all distinct and $m_i$, $n_j$ are positive integers, unless $\partial c = 0$. Define $\mathcal{F}(c) = \{d: d$ is a 1-chain on $R$ and $\partial d = \partial c\}$ and $\mathcal{H}(c) = \{d : d \in \mathcal{F}$ and $c - d$ is homologous to 0}. Consider fixed local coordinates $w_i$, $z_j$ defined in a neighborhood of $a_i$, $b_j$ respectively. Given vectors $r = (r_1, \cdots, r_J)$ and $s = (s_1, \cdots, s_J)$ of positive numbers, let $R(r, s)$ be the bordered Riemann surface obtained by removing from $R$ disks of radius $r_i$, $s_j$ about $a_i$, $b_j$.


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relative to these local coordinates. Set \( \mathcal{F}(r, s) = \{ d \cap R(r, s) : d \in \mathcal{F} \} \) and
\[
\lambda(\mathcal{F}) = \lim_{r, s \to 0} \lambda(\mathcal{F}(r, s)) + \frac{1}{2\pi} \left( \sum_{i=1}^{I} m_i^2 \log r_i + \sum_{j=1}^{J} n_j^2 \log s_j \right).
\]

\( \lambda(\mathcal{F}) \) exists and is called the reduced extremal length of the family \( \mathcal{F} \) with respect to the local coordinates \( w_i, z_j \). This quantity depends upon the choice of local coordinates in such a way that
\[
\exp(-2\pi \lambda(\mathcal{F})) \prod |dw_i|^{m_i} \prod |dz_j|^{n_j}
\]
is an invariant form. \( \lambda(\mathcal{H}) \) is defined in a similar fashion.

We also associate two singular differentials with \( c \). Let \( p \) be a harmonic function on \( R \) such that in a neighborhood of \( a_i, p = (n_i/2\pi) \log |w_i - a_i| + u_i \), where \( u_i \) is harmonic at \( a_i \), and near \( b_j, p = -(n_j/2\pi) \log |z_j - b_j| + v_j \), where \( v_j \) is harmonic at \( b_j \). The function \( p \) exists and is determined up to an additive constant. Set \( \psi_0 = \psi_0(\omega) = dp \) and \( \psi_\hbar = \psi_\hbar(\omega) = \psi_h - \psi_0 \), then for any \( \omega \in \Gamma_h(R) \), \( 0 = (\omega, \psi_\hbar) \) and \( \int_\omega \omega = (\omega, \psi_0) \). These inner products both exist since the integrals which give the inner products converge absolutely even though \( \psi_0 \) and \( \psi_\hbar \) have singularities. Set
\[
\langle \langle \psi_\hbar \rangle \rangle^2 = \lim_{r, s \to 0} \| \psi_\hbar \|^2_{\mathcal{F}(r, s)} + \frac{1}{2\pi} \left( \sum_{i=1}^{I} m_i^2 \log r_i + \sum_{j=1}^{J} n_j^2 \log s_j \right).
\]
This quantity exists but is not invariantly defined; however,
\[
\exp(-2\pi \langle \langle \psi_\hbar \rangle \rangle^2) \prod |dw_i|^{m_i} \prod |dz_j|^{n_j}
\]
is an invariant form. \( \langle \langle \psi_0 \rangle \rangle^2 \) is defined analogously. It can be shown that
\[
\langle \langle \psi_\hbar \rangle \rangle^2 - \langle \langle \psi_0 \rangle \rangle^2 = \| \psi_h \|^2.
\]

The following theorem is our main result.

**Theorem.** \( \lambda(\mathcal{F}(c)) = \langle \langle \psi_0(c) \rangle \rangle^2 \) and \( \lambda(\mathcal{H}(c)) = \langle \langle \psi_h(c) \rangle \rangle^2 \).

**Corollary.** \( \| \psi_h(c) \|^2 = \lambda(\mathcal{H}(c)) - \lambda(\mathcal{F}(c)) \).

This corollary leads to an extremal length interpretation of Abel's theorem. Let \( D \) be a divisor on the compact Riemann surface \( R \). Assume that either \( D=0 \) or \( D=B-A \), where \( A \) and \( B \) are disjoint integral divisors; that is, \( A = \sum_{i=1}^{I} m_i a_i \) and \( B = \sum_{j=1}^{J} n_j b_j \), the points \( a_i, b_j \) all being distinct and \( m_i, n_j \) being positive integers. \( D \) is called a principal divisor if there is a rational function \( f \) on \( R \) such that the divisor of \( f \) is \( D \). Abel's theorem asserts that \( D \) is a principal divisor if and only if there is a 1-chain \( c \) on \( R \) with the property that \( \partial c = D \) and \( \int_c \omega = 0 \) for all \( \omega \in \Gamma_h(R) \). Now, in order that \( \int_c \omega = 0 \) holds for all \( \omega \in \Gamma_h(R) \), it is necessary and sufficient that \( \| \psi_h(c) \|^2 = 0 \). Consequently, the next theorem has been established.
THEOREM. A divisor $D$ on a compact Riemann surface $R$ is principal if and only if there is a 1-chain $c$ on $R$ with $\partial c = D$ and $\lambda(F(c)) = \lambda(H(c))$.

Our main theorem has several analogs on an open Riemann surface. In fact, on an open surface there are six curve families associated with a 1-chain $c$. The reduced extremal length of all six families can be expressed in terms of singular differentials which are closely related to various reproducing differentials connected with $c$. By making use of these results, we can give an extremal length interpretation for the squared norm of the $\Gamma_h(R) \cap \Gamma^*(R)$-reproducing differential for a 1-chain $c$; here $x$ and $y$ can represent any one of $h$, $hse$, $ho$, $he$, $hm$, except that $x=y=he$ or $ho$ is not permitted. There are fourteen nontrivial such subspaces of $\Gamma_h(R)$.

REFERENCES


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