A SEPARABLE SOMEWHAT REFLEXIVE BANACH SPACE WITH NONSEPARABLE DUAL

ROBERT C. JAMES

Communicated by Jacob Feldman, November 18, 1973

ABSTRACT. An example is given of a separable Banach space \( X \) whose dual is not separable, but each infinite-dimensional subspace of \( X \) contains an infinite-dimensional subspace isomorphic to Hilbert space. Thus \( X \) contains no subspace isomorphic to \( c_0 \) or \( l_1 \), \( X \) is somewhat reflexive, and no nonreflexive subspace has an unconditional basis.

It has been conjectured that every infinite-dimensional Banach space has an infinite-dimensional subspace that is either reflexive or isomorphic to \( c_0 \) or to \( l_1 \) [9, p. 165]. A counterexample would also be an example of a space that has no infinite-dimensional subspace with an unconditional basis [6, Theorem 2, p. 521]. It is known that there is a nonreflexive Banach space \( J \) with no subspace isomorphic to \( c_0 \) or to \( l_1 \) [6, pp. 523–527], but \( J^{**} \) is separable. Each of the following is a necessary and sufficient condition for a separable Banach space \( X \) to contain a subspace isomorphic to \( l_1 \); separability is not needed for conditions (i) and (ii) (see [5, Theorem 2.1, p. 13] and [10, p. 475]).

(i) \( L_1[0, 1] \) is isomorphic to a subspace of \( X^* \).
(ii) \( C[0, 1]^* \) is isomorphic to a subspace of \( X^* \).
(iii) \( X^* \) has a subspace isomorphic to \( l_1(\Gamma) \) for some uncountable \( \Gamma \).

A natural and well-known conjecture in view of the preceding is that a Banach space has a subspace isomorphic to \( l_1 \) if the space is separable and its dual is not separable (e.g., see [1, §9, p. 243], [2, §5.4, p. 174], and the last paragraph of [11]). It will be shown that this conjecture is false. The counterexample \( \mathcal{X} \) has the property that each infinite-dimensional subspace has an infinite-dimensional subspace isomorphic to Hilbert space. Thus \( \mathcal{X} \) is also a counterexample to the conjecture that each separable somewhat-reflexive space has a separable dual (see [3, Problem 3, p. 191] and [8, Remark IV.2, p. 86]). Also, neither \( c_0 \) nor \( l_1 \) has an infinite-dimensional subspace isomorphic to Hilbert space, so no nonreflexive subspace has an unconditional basis [6, Theorem 2, p. 521]. It has been


1 Supported in part by NSF Grant GP-28578.

Copyright © American Mathematical Society 1974
shown by J. Lindenstrauss and C. Stegall that $X$ is a counterexample for several other conjectures. They will present these results in a later paper, as well as giving another example of a separable space with nonseparable dual that has no subspace isomorphic to $l_1$ (this space has a subspace isomorphic to $c_0$).

The counterexample is intimately related to the space $J$ mentioned above, e.g., many complemented subspaces are isometric to $J$. This follows from the fact that, if $x = \{x_i\} \in J$, then

$$\|x\| = \sup \left\{ \left[ \sum_{i=1}^{n-1} (x_{p(i+1)} - x_{p(i)})^2 \right]^{1/2} : 1 \leq p(1) < \cdots < p(n), n \geq 1 \right\},$$

but if $x$ is written as $\{\xi_i\}$, where $\xi_i = x_i - x_{i+1}$ and $x = \sum_{i}^{\infty} \xi_i e_i$ with $e_n$ the sequence for which $x_i = 1$ if $i \leq n$ and $x_i = 0$ if $i > n$, then

$$\|x\| = \sup \left\{ \left[ \sum_{i=1}^{n-1} \left( \sum_{j=p(i)}^{p(i+1)-1} \xi_i \right)^2 \right]^{1/2} : 1 \leq p(1) < \cdots < p(n), n \geq 1 \right\}.$$

To describe the counterexample, first choose a set $\Omega$ of cardinality $c$, each of whose members can be thought of as an infinite subset of the positive integers or as a nested sequence of intervals obtained as follows. Associate $1 = b(1, 1)$ with the interval $[0, 1]$, then $2 = b(2, 1)$ with $[0, \frac{1}{2}]$ and $3 = b(2, 2)$ with $[\frac{1}{2}, 1]$, and in general for each positive integer $n$ use the integers from $2^{n-1}$ to $2^n - 1$, or $\{b(n, i) : 1 \leq i \leq 2^{n-1}\}$, to label the $2^{n-1}$ intervals remaining at the $n$th stage of the process customarily used to describe the Cantor set. Now let each number $t$ in the Cantor set determine a member of $\Omega$, namely, the set of all integers associated with intervals containing $t$. By a segment we shall mean a finite increasing sequence (possibly empty) of consecutive members of some set $\mathcal{A} \in \Omega$. If $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ are in $\Omega$, then $\mathcal{A} \cap \mathcal{B}$ is a nonempty initial segment of both $\mathcal{A}$ and $\mathcal{B}$. A branch point of order $k$ for $\Omega$ is one of the integers $\{b(k, i) : 1 \leq i \leq 2^{k-1}\}$ that is the $k$th term of some member of $\Omega$. A branch of order $k$ (or a $k$-branch) is an infinite increasing sequence of consecutive members of some set $\mathcal{A} \in \Omega$ whose first member is a branch point of order $k$. For each sequence $x = \{x_i\}$ of real numbers with finite support, let

$$\|x\| = \sup \left\{ \left[ \sum_{n} \left( \sum_{i \in A(n)} x_i \right)^2 \right]^{1/2} \right\},$$

where the sup is over all finite sets $\{A(n) : 1 \leq n \leq p\}$ of pairwise disjoint segments. Let $X$ be the completion with respect to this norm of the normed linear space $X$ of all such sequences. Then $X$ is a separable Banach space.

For each $\mathcal{A} \in \Omega$, define a linear functional $f_{\mathcal{A}}$ on $X$ by letting $f_{\mathcal{A}}(x) = \sum_{i \in \mathcal{A}} x_i$ if $x \in X$, and extending to $X$ by use of continuity. Then $\|f_{\mathcal{A}}\| = 1$. 
Also, if \( m \in \mathcal{A} - \mathcal{B} \), \( n \in \mathcal{B} - \mathcal{A} \), and \( x \) is the sequence \( \{x_i\} \) with \( x_{m+1} = 1 \), \( x_n = -1 \) and \( x_i = 0 \) otherwise, then \( (f_{m+1,n} - f_{m,n})(x) = 2 \) and \( \|x\| = 2^{1/2} \). Thus \( \|f_{m+1,n} - f_{m,n}\| \geq 2^{1/2} \) and \( \mathcal{X}^* \) is not separable.

**Theorem.** If \( \theta > \sqrt{2} \), then each infinite-dimensional subspace of \( \mathcal{X} \) contains an infinite-dimensional subspace \( \mathcal{H} \) for which there is an inner-product norm \( \| \cdot \| \) such that

\[
\|x\| \leq \|x\| \leq \theta \|x\| \quad \text{if} \ x \in \mathcal{H}.
\]

**Proof.** It is sufficient to prove the theorem for \( \mathcal{X} \). Let \( \mathcal{Y} \) be an infinite-dimensional subspace of \( \mathcal{X} \) and let \( \mathcal{Y}^k \) be the subspace of \( \mathcal{Y} \) whose members are zero at each of the finite set of branch points with order less than \( k \). Then \( \mathcal{Y}^k \) has finite codimension as a subspace of \( \mathcal{X} \). For each \( x \) in \( \mathcal{X} \), let

\[
[x]_k = \sup \left\{ \left( \sum_{\ell \in \mathcal{B} \cap (m, n)} x_{m}^\ell \right)^{1/2} \right\},
\]

where the sup is over all sets \( \{\mathcal{B} \cap (m, n)\} \) of pairwise disjoint \( k \)-branches. Let

\[
\omega = \lim \inf_{k \to \infty} \{[x]_k : x \in \mathcal{Y}^k \text{ and } \|x\| = 1\}.
\]

It will be shown that \( \omega = 0 \). Suppose \( \omega > 0 \). For \( \varepsilon > 0 \), choose \( K \) so that

\[(1) \quad \inf \{[x]_k^2 : x \in \mathcal{Y}^k \text{ and } \|x\| = 1\} > \omega^2 - \varepsilon \quad \text{if} \ k \geq K.
\]

Choose an increasing sequence of integers \( \{m(k)\} \) with \( m(1) = K \), and then a sequence \( \{y^k\} \) in \( \mathcal{X} \) such that, for each \( k \), \( \|y^k\| = 1 \), \( y^k \) has nonzero terms only at branch points with orders in the interval \( [m(k) - 1, m(k)] \), and

\[(2) \quad [y^k]_{m(k)}^2 < \omega^2 + \varepsilon.
\]

It will be shown that a contradiction is obtained if \( \varepsilon \) is sufficiently small. Let \( y^k = \{y^k_i\} \). Since \( [y^k]_i^2 \geq \omega^2 - \varepsilon \) and \( y^k \in \mathcal{Y}^{m(k)} \), there are \( 2^{K-1} \) branch points of order \( m(k) \), which will be denoted by \( \{b(k, p_i^k) : 1 \leq i \leq 2^{K-1} \} \) rather than using \( b(m(k), p_i^k) \), and \( 2^{K-1} \) branches \( \{B(k, p_i^k) : 1 \leq i \leq 2^{K-1} \} \) of order \( m(k) \) starting at these branch points, such that

\[(3) \quad \sum_{i} \left( \sum_{\ell \in B(k, p_i^k)} y^k_{\ell} \right)^2 > \omega^2 - \varepsilon.
\]

Now for each \( i, k \) and \( \kappa \) with \( i \leq 2^{K-1} \) and \( \kappa < \kappa \), let \( \sigma(k, \kappa; i) \) be \( j \) if there exists \( j \leq 2^{K-1} \) such that \( b(k, p_j^k) \) and \( b(\kappa, p_j^\kappa) \) are on the same \( K \)-branch. Then \( \sigma(k, \kappa; i) \) is strictly increasing as a function of \( i \) and determines a one-to-one mapping of a subset of \( \{p_i^k : i \leq 2^{K-1} \} \) onto a subset of \( \{p_i^\kappa : i \leq 2^{K-1} \} \). Choose a sequence of positive integers \( I_1 \) so that if \( k \) and \( \kappa \) are in \( I_1 \) and \( \kappa < \kappa \), then \( \sigma(k, \kappa; i) = \sigma(k; i) \) is independent of \( \kappa \) for each \( i \).
Then choose a subsequence $I_2$ of $I_1$ so that if $k \in I_2$ then $\sigma(k; i) = \sigma(i)$ is independent of $k$ for each $i$. Now, $\sigma[\sigma(i)] = \sigma(i) = i$ and, for each $i \leq 2^{K-1}$, either $i$ is in the domain of $\sigma$ and there is a $K$-branch that contains all $b(k, p_i^k)$ for $k \in I_2$, or $i$ is not in the domain of $\sigma$ and no $K$-branch that contains $b(k, p_i^k)$ for some $k \in I_2$ can contain any $b(\kappa, p_i^k)$ for $\kappa \neq k$ and $\kappa \in I_2$.

Now choose a subsequence $I_3$ of $I_2$ such that, for each $i$ in the domain of $\sigma$ and any two members $k$ and $\kappa$ of $I_3$,

\[
\left| \sum_{i \in B} y_i^k - \sum_{i \in B} y_i^\kappa \right| < 2^{-K/2} \varepsilon,
\]

where $B$ is the $K$-branch containing all $b(k, p_i^k)$ for $k \in I_3$. For a $\lambda$ to be chosen later, let $\{\mu(j): 1 \leq j \leq \lambda\}$ be any $\lambda$ consecutive members of $I_3$ and, for any $K$-branch $B$, consider

\[
\left[ \sum_{i \in B} \left( \frac{1}{2} \right)^j y_i^{\mu(j)} \right]^2 = \left[ \sum_{i \in B} \left( \frac{1}{2} \right)^j \left( \sum_{i \in B} y_i^{\mu(j)} \right) \right]^2.
\]

For each $\mu(j)$, let $\sum_{i \in B} y_i^{\mu(j)}$ be denoted by $\rho_B^{\mu(j)}$ or $\Delta_B^{\mu(j)}$ accordingly as $B$ contains one of the branch points $\{b[\mu(j), p_i^{\mu(j)}]\}$ or $B$ does not contain any such branch point. Then either there exists $\iota \leq 2^{K-1}$ and $\kappa > 0$ such that $B$ contains $\{b(\mu(j), p_i^{\mu(j)}): j \leq \kappa\}$ and $B$ contains no other $b(\mu(j), p_i^{\mu(j)})$ for $\kappa < j \leq \lambda$ and $i \leq 2^{K-1}$, or else $B$ contains at most one of $\{b[\mu(j), p_i^{\mu(j)}]: 1 \leq j \leq \lambda, i \leq 2^{K-1}\}$. For any real numbers $\{a_i\}$,

\[
\left( \sum_{i=1}^{n} a_i \right)^2 \leq \sum_{i=1}^{n} 2^a_i^2.
\]

Therefore it follows from (4) that the expression (5) is not greater than

\[
2(\rho_B^\iota)^2 + 4 \left( \frac{K}{2} \right) 2^{-K/2} \varepsilon^2 + \sum_{j=1}^{\lambda} e_j 2^j (\Delta_B^{\mu(j)})^2,
\]

where $\mu \in \{\mu(j): j \leq \lambda\}$ (except that the first term in (7) may be missing), $\kappa$ is the largest integer such that $\kappa \leq \lambda$ and $B$ contains $b(\mu(j), p_i^{\mu(j)})$ for some $i$ and for all $j \leq \kappa$, and each $e_j$ is 0 or 1. Note that if we sum terms of type $(\Delta_B^{\mu(j)})^2$ over any $2^{K-1}$ pairwise disjoint $K$-branches $B(n)$, then it follows from (2) and (3) that this sum is not greater than $2\varepsilon$; also, there are then at most $2^{K-1}$ terms of the type of the first term in (7), so these can contribute to $[\sum_{j=1}^{\lambda} (\Delta_B^{\mu(j)})^2]$ for at most $2^{K-1}$ values of $j$. Therefore the sum of (5) or (7) over any $2^{K-1}$ pairwise disjoint $K$-branches is not greater than

\[
2 \cdot 2^{K-1} (\omega^2 + \varepsilon) + 4[\lambda/2] \varepsilon^2 + \lambda \cdot 2^\lambda (2\varepsilon) \leq 2^K \omega^2 + \varepsilon(2^K + \lambda \cdot 2^{\lambda+1}) + 2\lambda e_3.
\]
Since \( \| \sum_{i=1}^{4} (-1)^{i} y^{w(i)} \|^{2} \geq \lambda \), this contradicts (1) if

\[
2^{K} \omega^{2} + \epsilon(2^{K} + \lambda \cdot 2^{\lambda+1}) + 2\lambda\epsilon^{2} < \lambda(\omega^{2} - \epsilon).
\]

This inequality can be satisfied by choosing \( \lambda > 2^{K} \) and then choosing \( \epsilon \) small enough.

This concludes the proof that \( \omega = 0 \). Since \( \omega = 0 \), we can let \( \epsilon \) be a positive number and choose an increasing sequence of integers \( \{n(k)\} \) and a sequence \( \{y^{k}\} \) in \( X \), such that, for each \( k \), \( \|y^{k}\| = 1 \), \( y^{k} \) has nonzero terms only at branch points with orders in the interval \( (n(k), n(k+1)) \), and

\[
[y^{k}]_{n(k)}^{2} < 2^{-k}\epsilon^{2}.
\]

Let \( \{a_{i}\} \) be a finite sequence of real numbers with \( \sum a_{i}^{2} > 0 \). Then \( \| \sum a_{i}y^{j} \|^{2} \geq \sum a_{i}^{2} \). Choose a finite set \( \{A(n) : 1 \leq n \leq p\} \) of pairwise disjoint segments such that

\[
\| \sum a_{i}y_{i} \|^{2} = \sum_{n} \left( \sum_{i \in A(n)} \sum a_{i}y_{i} \right)^{2}.
\]

If \( A \) is any of these segments, then \( A \) is the union of an initial and a terminal segment, each of which contains a part of the piece of a branch between branch points of order \( n(j) \) and branch points of order \( n(j+1) \) for some \( j \), and several interior segments, each of which has the property that there is a \( j \) such that the segment contains all of the piece of a branch between branch points of order \( n(j) \) and branch points of order \( n(j+1) \). A sum \( \sum_{i} a_{i}y_{i}^{j} \) over those \( i \) in an initial segment or a sum over those \( i \) in a terminal segment contributes only to the norm of the corresponding \( a_{i}y_{i}^{j} \), while a sum over an interior segment contributes to \( [a_{i}y_{i}^{j}]_{n(i)} \) only. Now we can use the fact that

\[
(a + b + c)^{2} \leq (2 + \epsilon)(a^{2} + b^{2}) + (1 + 2\epsilon)c^{2}
\]

for any real numbers \( a, b, c, \) and then (6) and (8), to obtain

\[
\| \sum a_{i}y_{i} \|^{2} \leq (2 + \epsilon) \sum a_{i}^{2} + \left( 1 + \frac{2}{\epsilon} \right) \sum_{j} 2^{j}a_{j}^{2}[y_{j}]_{n(i)}^{2} < (2 + \epsilon) \sum a_{i}^{2} + (2\epsilon + \epsilon^{2}) \sum_{j} a_{j}^{2} = (2 + 3\epsilon + \epsilon^{2}) \sum a_{j}^{2}.
\]

Since \( \epsilon \) was arbitrary, for any \( \theta > \sqrt{2} \) there is an infinite sequence \( \{y^{k}\} \) of members of \( X \) such that, for all sequences \( \{a_{i}\} \) of real numbers,

\[
(\sum_{1}^{\infty} a_{i}^{2})^{1/2} \leq \| \sum_{1}^{\infty} a_{i}y_{i} \| \leq \theta \left( \sum_{1}^{\infty} a_{i}^{2} \right)^{1/2}.
\]
ERRATUM ADDED IN PROOF. The sequences \{m(k)\} and \{y^k\} should be chosen simultaneously so that each \(y^k\) is in \(Y\) and the branches \(\{B(k, y^k)\}: 1 \leq i \leq 2^{K-1}\) are pieces of pairwise disjoint \(K\)-branches; \(\mu(j): 1 \leq j \leq \lambda\) should not be consecutive members of \(I_3\), but chosen so that, for each branch point \(b\) of order \(K\),

\[
\sup \left\{ \left[ \sum_{i \in B} y^\mu(i) \right]^2 \right\} < \sup \left\{ \left[ \sum_{i \in B} y_i^\mu(1) \right]^2 \right\} + 2^{1-K} \varepsilon,
\]

where \(1 \leq j \leq \lambda\) and \(B\) is any \(K\)-branch containing \(b\). In the first term of (7), \(\rho^\mu_B\) should be replaced by the sum of the absolute values of two such terms; in the next two inequalities, \(2^K(\omega^2 + \varepsilon)\) can now be replaced by \(8\omega^2 + 16\varepsilon\) and \(\lambda\) need not depend on \(K\).

REFERENCES


DEPARTMENT OF MATHEMATICS, CLAREMONT GRADUATE SCHOOL, CLAREMONT, CALIFORNIA 91711