A MEROMORPHIC FUNCTION WITH ASSIGNED NEVANLINNA DEFICIENCIES

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1. Statement of result.

THEOREM. Let \( \delta(a) \) and \( \theta(a) \) be nonnegative functions defined on the extended complex plane \( \hat{\mathbb{C}} \) such that \( 0 \leq \delta(a) + \theta(a) \leq 1, \ a \in \hat{\mathbb{C}}, \)

\[
\sum_{a \in \hat{\mathbb{C}}} (\delta(a) + \theta(a)) \leq 2.
\]

Then there exists a function \( f(z) \) which is meromorphic in the finite \( \mathbb{C} \)-plane with \( \delta(a, f) = \delta(a), \ \theta(a, f) = \theta(a), \ a \in \hat{\mathbb{C}}. \) Finally, let \( \phi(r) \) be a positive increasing function with

\[
(1.1) \quad \phi(r) \to \infty \quad (r \to \infty).
\]

Then our function \( f(z) \) may be chosen so that, in addition, its Nevanlinna characteristic satisfies

\[
(1.2) \quad T(r) \leq r^{\phi(r)}
\]

for all sufficiently large \( r. \)

Here we are using the standard notations of Nevanlinna's theory as described in [3, 6]; for example, \( \theta(a, f) \) is the index of multiplicity (Verzweigungsindex) of \( a. \) Our function \( f(z) \) thus provides a complete solution to the 'inverse problem' of the Nevanlinna theory (cf. [2, Chapter 7]; [9, Chapter 8]).

In general, the solution to the inverse problem must be of infinite order (cf. [8]); (1.2) asserts that this may be as 'small' an infinite order as desired.

Among earlier partial solutions to this problem we note Nevanlinna [5], Goldberg (cf. [2, Chapter 7, Theorems 8.2, 8.3]) and Fuchs-Hayman (cf. [3, §4.1]).

2. Method of proof. Given the function \( \phi(r) \) of (1.1) and the sets \( \Delta = \{a; \delta(a) > 0\} \) and \( \Theta = \{a; \theta(a) > 0\}, \) we shall associate a sequence \( \{r_k\} \)

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with \( r_{k+1}/r_k \) tending rapidly to infinity with the property that if \( b \) is a fixed element of \( \mathcal{C} - (\Delta \cup \theta) \) then, for all \( a \in \mathcal{C} \),

\[
(2.1) \quad |(1 - \delta(a))n(r, b, f) - n(r, a, f)| \leq (4/k)n(r, b, f) \quad (r_k \leq r \leq r_{k+1}),
\]

\[
(2.2) \quad |\theta(a)n(r, b, f) - \{n(r, a, f) - \bar{n}(r, a, f)\}| < (4/k)n(r, b, f) \quad (r_k \leq r \leq r_{k+1})
\]

and

\[
(2.3) \quad 2^k(1 - k^{-1}) \leq n(2r, b, f)/n(r, b, f) \leq 2^{k+1}(1 + k^{-1}) \quad (r_k \leq r \leq r_{k+1}).
\]

That \( f \) has preassigned deficiencies and indices of multiplicity follows from (2.1), (2.2) and the fact (cf. [6, p. 276]) that there is a set \( E \) of logarithmic capacity zero such that

\[
N(r, a, f) \sim T(r, f) \quad (r \to \infty, a \notin E).
\]

Also, if \( r_{k+1}/r_k \) increases sufficiently rapidly, (2.3) shows that the growth of \( T(r, f) \) may be retarded in accord with (1.2).

One first constructs a ‘quasi-meromorphic’ function\(^{2} \) \( g(z) \) which satisfies, formally, (2.1), (2.2) and (2.3), and then factors

\[
(2.4) \quad g = f \circ h
\]

where \( f \) is a meromorphic function and \( h \) is a quasiconformal homeomorphism of the complex plane onto itself. The problem is to ensure that \( h \) in (2.4) sufficiently approximates the identity so that (2.1), (2.2) and (2.3) (with, perhaps, a different sequence \( \{r_k\} \)) transfer to \( f \).

Using an important principle of Teichmuller [7], Le Van Thiem [4] first applied this principle to the inverse problem, and the method was further exploited by Goldberg (cf. [2, Chapter 7]). These efforts had two limitations: the characteristic of \( g \) had to be of finite order and the dilatation of \( g \), \( d_g(z) = |g_z(z)/g_z(z)| \) was subject to

\[
(2.5) \quad \int_{|z| \geq 1} |d_g(z)|^2 \, dx \, dy < \infty.
\]

In [1, Theorem 2], it was shown that this principle applies under the more flexible condition

\[
(2.6) \quad D_g(r) \equiv \int_0^{2\pi} d_g(re^{i\theta}) \, d\theta = o(1) \quad (r \to \infty),
\]

\(^{2} \) A ‘quasi-meromorphic function’ is one which may be expressed as in (2.4).
and the freedom allowed by (2.6) is decisive here. For it is not hard to show that \( f \) and \( g \) will have the same deficiencies and indices of multiplicity if \( D_g(r) \) decreases very rapidly with respect to \( \log(n(2r, b, g)/n(r, b, g)) \).

By increasing the ratios \( r_{k+1}/r_k \) we can diminish \( D_g(r) \) with respect to \( \log(n(2r, b, g)/n(r, b, g)) \); thus (2.5) is very unlikely to hold. Finally, \( g \) is constructed by piecing together functions discussed in [1] and [5].

ADDED IN PROOF (MARCH 15, 1974). Dr. A. A. Goldberg has informed me that the substitution of (2.6) for (2.5) first appears in the work of P. P. Belinskii. *The behavior of quasiconformal mappings at an isolated singular point*, Učen. Zap. L’vov. Gos. Univ. 29 (1954), 58–70. (Russian). However, Belinskii did not apply this to the inverse problem.

REFERENCES


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