A CONDITIONAL LOCAL LIMIT THEOREM AND ITS APPLICATION TO RANDOM WALK

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1. Introduction. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. (independent and identically distributed) random variables defined on a probability space \((\Omega, F, P)\). In all that follows we assume that the \( X_i \) are distributed on the lattice of integers with \( EX_i = 0 \) and \( EX_i^2 = \sigma^2 < \infty \). For the recurrent random walk \( S_n \) with \( S_0 = 0 \) and \( S_n = X_1 + \cdots + X_n \) for \( n \geq 1 \), define the stopping time \( T \) either to be the first time \( S_n \) returns to zero or to be \( +\infty \) if no such \( n \) exists. We shall assume further that the random walk \( S_n \) is aperiodic. It is well known that \( T \) is finite with probability one and that \( n^{1/2}P[T > n] \) converges to the limit \((2/\pi)^{1/2} \sigma^{-1/2} \) as \( n \) approaches infinity. It follows from a result of Kesten [4] that \( n^{3/2}P[T = n] \) has limit \( \sigma/(2\pi)^{1/2} \) as \( n \) approaches infinity. In this paper we consider the asymptotic behavior of random walks conditioned by the events \([T > n]\) and \([T = n]\). Belkin [1] has obtained the result

\[
\lim_{n \to \infty} P[S_n/n^{1/2} \leq x \mid T > n] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} dy.
\]

We obtain a local limit theorem which is readily seen to be a generalization of this result. Our local version is then applied to obtain the weak convergence of a sequence of probability measures on \( C[0, 1] \) corresponding to a random walk conditioned by the event \([T = n]\). The limiting probability measure corresponds to a Markov process first introduced by Lévy [5] and subsequently entitled a Brownian excursion by Itô and McKean [3].

2. A conditional local limit theorem. Our main result is stated as

**Theorem 1.** Suppose the random variables \( X_1, X_2, \ldots \) are i.i.d. on the lattice of integers with \( EX_i = 0 \) and \( EX_i^2 = \sigma^2 < \infty \). Then

\[
\lim_{n \to \infty} \sup_{x} |n^{1/2}P[S_n = x \mid T > n] - (|x|/2\sigma n^{1/2})e^{-x^2/(2n\sigma^2)}| = 0.
\]

For any integer \( x \) define the hitting time \( T_{(x)} \) either to be the first \( n \geq 1 \) such that \( S_n = x \) or to be \( +\infty \) if no such \( n \) exists. Employing Theorem 1

and the facts that $P[S_n=x; T>n]=P[T(x)=n]$ and $n^{1/2}P[T>n] \to (2/\pi)^{1/2} \sigma$ as $n \to \infty$, we obtain

**Corollary 1.** Under the hypotheses of Theorem 1,

$$
\lim_{n \to \infty} \sup_{x} \left| nP[T(x) = n] - \left( |x|/\sigma n^{1/2} \right) \phi(x/\sigma n^{1/2}) \right| = 0,
$$

where $\phi(t)$ denotes the standard normal probability density function.

3. The weak convergence of random walk conditioned by the event $[T=n]$. On $C[0,1]$ with the uniform norm and the corresponding sigma field $\mathcal{E}$ of Borel subsets, define a sequence of probability measures $\{P_n\}$ by assigning mass

$$
P[S_1/\sigma n^{1/2} = x_1, \ldots, S_n/\sigma n^{1/2} = x_n \mid T = n]
$$

to the polygonal line segment $\xi$ such that $\xi(0)=0$ and $\xi(k/n)=x_k$ for $k=0, 1, \ldots, n$.

As an application of Corollary 1 we obtain

**Theorem 2.** The sequence of probability measures $\{P_n\}$ on $(C[0,1], \mathcal{E})$ converges weakly to a probability measure $P$ which corresponds to the Brownian excursion stochastic process.

The Brownian excursion is a Markov process with nonstationary transition density. Itô and McKean [3] discuss two alternative derivations of this process and provide explicit expressions for the transition density. Belkin [2] previously has obtained results analogous to Theorem 2 with the conditioning event $[T>n]$.

Proofs of these results will appear elsewhere.

**References**


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