The purpose of this note is to announce certain results I have obtained about the behavior of the Harish-Chandra C-function as a meromorphic function. The notation and terminology, if not explained, are that of [2], [3], or [6].

1. The C-ring. Let \((P, A)\) be a fixed parabolic pair of a semisimple Lie group \(G\) having finite center, \(P=MAN\) the corresponding Langlands decomposition, \(K\) a fixed maximal compact subgroup. Let \(\mathcal{G}, \mathcal{H}, \mathcal{H}_M, \mathcal{M}\) be the universal enveloping algebras of \(G, K, K_M,\) and \(M,\) respectively \((K_M=K\cap M)\) — i.e. of their complexified Lie algebras \(\mathfrak{g}_C, \mathfrak{t}_C, \mathfrak{t}_{M, C}, \mathfrak{m}_C.\)

Let \(b \mapsto b^t\) \((b \in \mathcal{G})\) denote the unique anti-automorphism of \(\mathcal{G}\) such that \(X^t = -X\) \((X \in \mathfrak{g}).\) Consider \(\mathcal{H}\) to be a right \(\mathcal{H}_M\)-module via the multiplication in \(\mathcal{H} \cdot b \cdot d = bd\) \((b \in \mathcal{H}, d \in \mathcal{H}_M),\) and consider \(\mathcal{M}\) to be a left \(\mathcal{H}_M\)-module via the operation \(d \cdot c = cd^t\) \((d \in \mathcal{H}_M, c \in \mathcal{M}).\) We can then form the tensor product \(\mathcal{H} \otimes \mathcal{H}_M \mathcal{M}\) of \(\mathcal{H}_M\)-modules. (We write \(b \otimes c\) for elements of \(\mathcal{H} \otimes \mathcal{H}_M \mathcal{M}, b \otimes c\) for elements of \(\mathcal{H} \otimes \mathcal{M}.)\) The group \(K_M\) acts on \(\mathcal{H} \otimes \mathcal{H}_M \mathcal{M}\) via the (well-defined) representation \(\rho: \rho(m)(b \otimes c) = b^m \otimes c^m\) \((b \in \mathcal{H}, c \in \mathcal{M}, m \in K_M).\) Let \(\mathcal{H} \otimes \mathcal{H}_M \mathcal{M})^{K_M}\) denote the \(K_M\)-invariants.

Proposition 1. \((\mathcal{H} \otimes \mathcal{H}_M \mathcal{M})^{K_M}\) is a ring (i.e., the "obvious" multiplication is well defined). In fact, it is a left and right Noetherian integral domain (noncommutative, in general), hence has a quotient division algebra.

We refer to \((\mathcal{H} \otimes \mathcal{H}_M \mathcal{M})^{K_M}\) as the C-ring associated to the pair \((P, A).\)

Let \(\tau\) be a left or double representation of \(K\) on a finite-dimensional Hilbert space \(V.\) Then there exists a representation \(\lambda,\) of the ring \(\mathcal{H} \otimes \mathcal{M}\) on \(C^\infty (M; V)\) defined as follows:

\[
\lambda_c(b \otimes c)\psi(m) = \tau(b)\psi(cm) \quad (b \in \mathcal{H}, c \in \mathcal{M}, m \in M, \psi \in C^\infty (M; V)).
\]

Let \(C^\infty (M, \tau_M)\) denote the space of \(\psi \in C^\infty (M; V)\) such that

\[
\tau(k)\psi(m) = \psi(km) \quad (k \in K_M, m \in M)
\]

if \(\tau\) is a left representation of \(K\) or such that

\[
\tau(k_1)\psi(m)\tau(k_2) = \psi(k_1mk_2) \quad (k_1, k_2 \in K_M, m \in M)
\]
if \( \tau \) is a double representation of \( K \) on \( V \). Then the rule

\[
\lambda_i(\sum b_j \otimes c_j)\psi(m) = \sum \tau(b_j)\psi(c_jm) \quad (b_j \in K, \ c_j \in M)
\]

defines a representation of the \( C \)-ring \((K \otimes K_M)^{K_M} \) on \( C^\infty(M, \tau_M) \). Clearly the spaces \( \mathcal{C}(M, \tau_M) \) and \( \mathcal{C}^\infty(M, \tau_M) \) of Schwartz functions and cusp forms in \( C^\infty(M, \tau_M) \) respectively are invariant subspaces.

2. The difference equations satisfied by the \( C \)-function. By a polynomial function on a connected simply connected nilpotent Lie group \( N \), we mean a function \( f \in C^\infty(N) \) such that \( X \rightarrow f(\exp X) \ (X \in L(N)) \) is a polynomial function on the Lie algebra \( L(N) \) of \( N \).

By a semilattice \( L \) in a real vector space \( V \), we mean an additive semigroup generated by a basis of \( V \).

**Proposition 2.** There exists a semilattice \( L \subseteq \alpha^* \) such that \( \mu \in L \) implies that \( e^{2i\mu(H(\alpha))} \) is a polynomial function on \( N \).

**Theorem 1** (The difference equations). Let \( \mu \in \alpha^* \) be such that \( e^{2i\mu(H(\alpha))} \) is a polynomial function on \( N \). Then there exist polynomials \( b^\mu(v) \), \( c^\mu(v) \) with coefficients in the \( C \)-ring \( (\mathcal{H} \otimes \mathcal{H}_M \otimes K)^{K_M} \) such that, for all double unitary representations \( \tau \) of \( K \),

\[
\lambda_i(b^\mu(v))C_{P|P}(1: v) = \lambda_i(c^\mu(v))C_{P|P}(1: v - 2i\mu) \quad (v \in \alpha^*).
\]

The polynomials \( b^\mu(v) \) and \( c^\mu(v) \) have the same degree and the same leading term, which we may assume lies in \( C[v] \) (i.e., is a scalar polynomial). The coefficients of \( c^\mu(v) \), in fact, lie in the subring \( \mathcal{Z}_M \) (the center of \( M \)) of the \( C \)-ring. The operators \( \lambda_i(b^\mu(v)), \lambda_i(c^\mu(v)) \) are never identically zero (as polynomials in \( v \)).

Taking \( \mu = \mu_1, \ldots, \mu_t \) to be generators of a semilattice \( L \) as in Proposition 2, we get the result that the \( C \)-function \( C_{P|P}(1: v) \) satisfies a system of \( l = rkP \) linear first order partial difference equations with polynomial coefficients.

3. The asymptotic development.

**Theorem 2.** Choose \( \lambda \in \alpha^*_c \) such that \( \text{Re}(\lambda, \alpha) > 0 \) for all roots \( \alpha \) of the pair \( (P, A) \). Then there exists a formal power series \( \sum_{j=0}^\infty t^{-j}b_j^{(\lambda)}(v) \) with coefficients in \( (\mathcal{H} \otimes \mathcal{H}_M \otimes K)^{K_M} \otimes C[v] \) (depending analytically on \( \lambda \)) such that

1. \( b_0^{(\lambda)}(v) \in C \);
2. \( b_j^{(\lambda)}(v) \) is of degree at most \( 2j \) in \( v \) (\( j \geq 0 \)); and
3. for every double representation \( \tau \) of \( K \),

\[
C_{P|P}(1: v + it\lambda) \sim t^{-s/2} \sum_{j=0}^\infty t^{-j}\lambda_i(b_j^{(\lambda)}(v)) \text{ as } t \rightarrow \infty
\]
uniformly for \( v \) in compact subsets of \( \mathfrak{a}_C^* \) (both sides being considered as operators on the space \( \mathcal{C}(M, \tau_M) \)). This means that, for each integer \( n \geq 0 \),

\[
\lim_{t \to \infty} t^{s/2} C_{P|P}(1; v + it\lambda) = \sum_{j=0}^{n} t^{-j} \lambda_{s}(b_{j}(v)) = 0.
\]

Here \( s = \dim N \). Replacing \( \tau \) by the trivial representation of \( K \), we get the same asymptotic expansion for the integral \( C(v) = \int_{R} e^{it\rho(H(\delta))} \, d\lambda \).

**Corollary 1.** Suppose that \( \lambda \) is as in Theorem 2. Then there exists a constant \( \zeta_{2} \) such that

\[
\lim_{t \to \infty} t^{s/2} C_{P|P}(1; v + it\lambda) = \zeta_{2} \times \text{id}
\]
as an operator on \( \mathcal{C}(M, \tau_M) \), the limit being uniform in \( v \) on compact subsets of \( \mathfrak{a}_C^* \).

4. **The representation theorems.**

**Theorem 3.** Choose \( \mu \in \mathfrak{a}_C^* \) such that \( \langle \mu, \alpha \rangle > 0 \) for all roots \( \alpha \) of \( \langle P, A \rangle \) and \( e^{2\mu(H(\delta))} \) is a polynomial function on \( N \). Let \( b(v) = b_{\mu}(v) \), \( c(v) = c_{\mu}(v) \) be as in Theorem 1. Then

\[
C_{P|P}(1; v) = \text{const} \times \lim_{n \to \infty} n^{-s/2} \lambda_{s}(c(v + 2i\mu) \cdots c(v + 2in\mu))^{-1} \times \lambda_{s}(b(v + 2i\mu) \cdots b(v + 2in\mu))
\]
(the constant being independent of \( \tau \)).

**Theorem 4.** Let \( \langle P, A \rangle \) be an arbitrary parabolic subgroup of \( G \); and let \( \tau \) be a double unitary representation of \( K \). Then there exist \( \mu_1, \cdots, \mu_r \in \mathfrak{a}_C^* \) and constants \( p_{ij}, q_{ij} \) \( (i=1, \cdots, r, j=1, \cdots, j_i) \) depending on \( \tau \) such that

\[
\det C_{P|P}(1; v) = \text{const} \times \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma(-i\langle v, \alpha_i \rangle + \mu_i, \alpha_i)}{\Gamma(-i\langle v, \alpha_i \rangle + p_{ij})},
\]

where \( \alpha_1, \cdots, \alpha_r \) are the reduced roots of \( \langle P, A \rangle \).

**Open Question.** Are the numbers \( p_{ij}, q_{ij} \) always rational?

5. **Idea of the proofs.** Theorem 1 is based on the following sequence of results.

Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) containing \( \mathfrak{a} \); and let \( P_+ \) denote the set of roots \( \beta \) of \( \langle \mathfrak{g}_C, \mathfrak{h}_C \rangle \) such that \( \beta|\mathfrak{a} > 0 \). Let \( X_{\beta}, X_{-\beta} \) (\( \beta \in P_+ \)) be root vectors such that \( B(X_{\beta}, X_{-\beta}) = 1 \) and \( \theta(X_{\beta}) = -X_{-\beta} = -X_{\beta} \).

Define vector fields \( q(X) \) \( (X \in \mathfrak{g}) \) on \( \bar{N} = \theta(N) \) by the following rule:

\[
q(X)f(\bar{n}) = -\sum_{\beta \in P_+} B(X, X_{\bar{n}}^\beta) f(\bar{n}; X_{-\beta}) \quad (f \in C^{\infty}(\bar{N})).
\]
PROPOSITION 3. \( X \mapsto q(X) \) defines a representation of \( \mathfrak{g} \) by derivations of \( C^\infty(N) \). The ring \( \mathcal{R}_g \) of polynomial functions on \( N \) is a \( q \)-invariant subspace of \( C^\infty(N) \).

Let \( \sum_\theta (P, A) = \{ \alpha_1, \ldots, \alpha_l \} \) be the simple roots of \( (P, A) \); and choose \( H_j \in a \) such that \( \alpha_i(H_j) = \delta_{ij} \).

PROPOSITION 4. Suppose that \( X \in \mathfrak{g} \). Then

\[
e_s(X) x = \left( \sum_{j=1}^l \langle iv - \rho, \alpha_j \rangle B(X, H_j(x)) \right) e_s(x) \quad (x \in G).
\]

(\( B \) is the Killing form on \( \mathfrak{g} \); \( e_s(x) = e^{iv-\rho(H(x))} \).)

COROLLARY 2. Suppose that \( Z \in \mathfrak{t} \). Then

\[
q(Z) e_s(n) = \left( \sum_{j=1}^l \langle iv - \rho, \alpha_j \rangle B(Z, H_j) \right) e_s(n) \quad (n \in N).
\]

Let \( V_1, \ldots, V_r \) \((t=\dim m)\) be an orthonormal basis for \( m_{\mathfrak{c}} \) (with respect to the Killing form). Also, given \( \psi \in C^\infty(M, \tau_M) \), define \( \hat{\psi} : G \rightarrow C^\infty(M : V) \) by

\[
\hat{\psi}(x | m) = \psi(xm) \quad (x \in G, m \in M).
\]

PROPOSITION 5. Suppose that \( Z \in \mathfrak{t} \) and \( \psi \in C^\infty(M, \tau_M) \). Then

\[
\hat{\lambda}_r(Z \otimes 1) \hat{\psi}(n) = -q(Z) \hat{\psi}(n) - \sum_j B(Z, V_j) \hat{\lambda}_r(1 \otimes V_j) \hat{\psi}(n) \quad (n \in N).
\]

PROPOSITION 6. There exists a unique \( \mathcal{M} \otimes C[v] \) module homomorphism

\[
F : \mathcal{K} \otimes \mathcal{K}_M \mathcal{M} \otimes C[v] \rightarrow \mathcal{M} \otimes \mathcal{R} - \otimes C[v]
\]
such that

\[
\begin{align*}
(1) & \quad F(1) = 1; \\
(2) & \quad F(v | n)(Z) = \sum \langle iv + \rho, \alpha_j \rangle B(Z, H_j^n) - \sum B(Z, V_j^n) V_j \quad (Z \in \mathfrak{t}); \\
(3) & \quad F(Zb) = F(b) F(Z) + q(Z) F(b) \quad (Z \in \mathfrak{t}, b \in \mathcal{K}); \\
(4) & \quad F(b \otimes c) = c F(b) \quad (b \in \mathcal{K}, c \in \mathcal{M}).
\end{align*}
\]

PROPOSITION 7. Suppose that \( b \in \mathcal{K} \otimes \mathcal{K}_M \mathcal{M} \). Then there exists a constant \( C = C(b) > 0 \) such that if \( \Im \langle v, \alpha_j \rangle \geq C(b) (j = 1, \ldots, l) \), then

\[
\lambda_r(b) \int_S E_s(n) \psi(nm) d\bar{n} = \int_N E_s(n) \lambda_r(1 \otimes F(v | n)(b)) \hat{\psi}(n | m) d\bar{n} \quad (m \in M),
\]

both integrals being convergent.

PROPOSITION 8. Given \( \phi(n) \in \mathcal{R}_g \), we can find \( b(v) \in \mathcal{K} \otimes \mathcal{K}_M \mathcal{M} \otimes C[v] \) and \( c(v) \in \mathcal{M} \otimes C[v] \) such that \( F(b(v)) = c(v) \phi \).
PROOF OF THEOREM 1. First apply Proposition 8 with $\phi(\tilde{n})=e^{2\mu(H(\tilde{n}))}$. Then apply Proposition 7.

The following is the essential step in the proof of Theorem 2.

PROPOSITION 9. Suppose that $\nu \in \mathfrak{a}_C^*$ and let $f_\nu(H)(\tilde{n})=\nu(H(\tilde{n}))$ ($\tilde{n} \in \tilde{N}$).

1. First apply Proposition 8 with $\phi(\tilde{n})=e^{2\mu(H(\tilde{n}))}$.
2. Then apply Proposition 7.

Then apply Proposition 7. Furthermore if $\nu \in \mathfrak{a}_C^*$ and $\langle \nu, \alpha \rangle > 0$ for all $\alpha \in \Sigma (P, A)$, then the critical point of the (real-valued) function $f_\nu(H(\tilde{n}))$ has index 0.

Proposition 9 allows us to apply the method of steepest descent (see [1]) to derive the asymptotic expansion of $C_{P|P}(1:\nu)$ (Theorem 2).

Theorems 3 and 4 follow fairly easily, given Theorems 1 and 2.

6. An example: the $C$-function for the group $SU(1, 2)$. In this case, the set $P_+$ consists of three roots $\beta_1, \beta_2$ and $\beta_3$, where $\beta_1$ and $\beta_2$ are simple and $\beta_3=\beta_1+\beta_2$. Also, the parabolic pair $(P, A)$ is minimal; so the $C$-ring is isomorphic to $\mathcal{K} M$. If $\mu=\alpha$ (the simple root of $(P, A)$), $e^{2\mu(H(\tilde{n}))}$ is a polynomial function on $\tilde{N}$; the corresponding polynomials $b^\mu(\nu)$ and $c^\mu(\nu)$ are then as follows

$$b^\mu(\nu) = b^\mu_1(\nu)b^\mu_2(\nu),$$

where

$$b^\mu_1(\nu) = \{ (i\nu + \rho, \alpha) - i(\sqrt{6/6})Z_{\beta_1}(i\nu, \alpha) + \frac{1}{2}V + \frac{1}{2}Z_{\beta_1}Z_{\beta_2} \},$$

$$b^\mu_2(\nu) = \{ (i\nu + \rho, \alpha) - \frac{1}{2}V)(i\nu, \alpha) + i(\sqrt{6/6})Z_{\beta_2} + \frac{1}{2}Z_{\beta_1}Z_{\beta_2} \},$$

and

$$c^\mu(\nu) = (i\nu + \rho, \alpha)(i\nu + \alpha, \alpha)(i\nu + \rho, \alpha) + \frac{1}{2}V)(i\nu + \rho, \alpha) - \frac{1}{2}V).$$

Here $Z_{\beta_i}=\frac{1}{2}(X_{\beta_i}+\theta(X_{\beta_i}))$ ($X_{\beta_i}$ normalized as above), and $V$ is the element of $m_C$ such that $\beta_1(V)=\frac{1}{2}$.

Using the polynomials $b^\mu(\nu)$ and $c^\mu(\nu)$, we obtain the following result.

PROPOSITION 10. Let $\tau=\tau_{m,n}$ be the $(m+1)$-dimensional representation of $K=U(2)$ such that $\tau(V-i\sqrt{6}Z_{\beta_3})=n\times 1$. ($V-i\sqrt{6}Z_{\beta_3}$ spans the center of $\mathfrak{g}_C$.) Let $\mathcal{V}=\mathcal{V}^{(m,n)}$ denote the space of $\tau$. Then $\mathcal{V}$ has a basis $x_j$ ($j=0, 1, \cdots, m$) such that $\tau(V)x_j=\frac{1}{2}(m+n-2j)x_j$. Furthermore, the operator

$$C^{(m,n)}_{P|P}(1:\nu) = \int_{\tilde{N}} \tau(k(\tilde{n}))e^{i\nu-H(\tilde{n})} d\tilde{n}$$

on $\mathcal{V}^{(m,n)}$ has each vector $x_j$ as an eigenvector; and the corresponding eigenvalue is

$$\frac{2}{\sqrt{\pi}} \frac{\Gamma(\xi_4)\Gamma(\xi_5)\Gamma(\xi_6)\Gamma(\xi_4)}{\Gamma(\xi_5)\Gamma(\xi_6)\Gamma(\xi_7)\Gamma(\xi_8)}.$$
where \( \zeta_1 = \omega - i \langle \nu, \alpha \rangle / 2 \sigma(\alpha, \alpha) \), \( \zeta_2 = \zeta + \frac{1}{2} \), \( \zeta_3 = \zeta + \frac{3}{2} \nu - \frac{1}{2} m - \frac{3}{2} n \), \( \zeta_4 = \zeta - \frac{3}{2} \nu + \frac{1}{2} m + \frac{3}{2} n \), \( \zeta_5 = \zeta + \frac{1}{2} \nu - \frac{3}{2} m - \frac{1}{2} n \), \( \zeta_6 = \zeta + \frac{1}{2} \nu + \frac{1}{2} m - \frac{1}{2} n + 1 \), \( \zeta_7 = \zeta - \frac{3}{2} \nu - \frac{1}{2} m + \frac{1}{2} n \), and \( \zeta_8 = \zeta - \frac{1}{2} \nu + \frac{3}{2} m + \frac{1}{2} n + 1 \).

Detailed proofs of these results and some more examples will appear in a paper in preparation.

REFERENCES


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