AN EMBEDDING-OBSTRUCTION
FOR PROJECTIVE VARIETIES

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A classical problem in differential topology is the following: Let \( X \) be a compact \( n \)-dimensional differentiable manifold (without boundary). Then compute the least integer \( m=m(X) \) such that \( X \) may be embedded into \( \mathbb{R}^m \). Usually this question is attacked as follows (see Atiyah [1]): (a) An upper bound for \( m \) is obtained by exhibiting explicit embeddings, and (b) a lower bound is obtained by certain homotopy invariants.

The forthcoming paper [2] deals with an algebro-geometric counterpart to the problem mentioned above: Let \( X \) be a nonsingular, projective \( k \)-variety embedded in some projective space \( \mathbb{P}^N_k \) by the embedding \( i \). For simplicity we assume the field \( k \) to be algebraically closed, but the results of [2] still hold under the weaker assumption that \( k \) is infinite.

The main result is that the least integer \( m=m(X,i) \), such that \( X \) can be embedded into \( \mathbb{P}^m_k \) via a projection from \( \mathbb{P}^N_k \), is effectively computed in terms of the degrees of the Chern-classes of \( X \).

More precisely, let \( X \subset \mathbb{P}^N_k \) be an \( n \)-dimensional nonsingular projective variety, embedded in \( \mathbb{P}^N_k \). Let \( c_i=c_i(X)=c_i(O_{\mathbb{P}^N_k}) \in A(X) \) be the Chern-classes of \( X \), where \( A(X) \) denotes the Chow-ring of \( X \). Consider the formal inverse of the alternating Chern-polynomial:

\[
\left[ \sum_{i=0}^n (-1)^i c_i T^i \right]^{-1} = \sum_{i=0}^\infty f_i T^i.
\]

Here \( f_i=0 \) for \( i>n \). Let \( d_i=\text{deg}(f_i) \) with respect to the embedding \( i:X \subset \mathbb{P}^N_k \). In particular \( d_0=\text{deg}(i(X))=d \). Define

\[
B_X(T) = \left( \sum_{i=0}^n d_i T^i \right) \left( \sum_{i=0}^{2n+1} \binom{2n+2}{i} T^i \right) = B_0 + B_1 T + \cdots,
\]

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of which we only need $B_0, B_1, \cdots, B_n$. In fact, we put

$$\beta_j = \sum_{i=0}^{i=n} (-1)^i \binom{n-i}{j-i-n} \left( B_i - d^2 \binom{n+1}{i} \right), \quad n \leq j \leq 2n,$$

$$\beta_j = 1 \quad \text{for } j < n, \quad \beta_j = 0 \quad \text{for } j > 2n.$$

**Definition.** For all integers $m$ the sequence $\left( \beta_m, \beta_{m+1}, \cdots \right)$ is called the $m$th embedding obstruction of the embedded variety $(X, i)$.

In [2] the following result is proved:

**Theorem.** If $m < N$, then $X$ can be embedded into $P_k^n$ via a projection from $P_k^N$ if and only if the $m$th obstruction vanishes, i.e.

$$(\beta_m, \beta_{m+1}, \cdots) = (0, 0, \cdots).$$

This implies at once the well-known and classical (see E. Lluis [5]):

**Corollary.** $m(X, i) \leq 2n+1$.

For $n=1$ and $m=2$ we obtain the well-known genus-formula

$$g(X) = \frac{1}{2} (d - 1)(d - 2)$$

which is necessary and sufficient for when the nonsingular curve $X$ can be projected isomorphically onto a plane curve. For $n=2$, $m=3$, we get that a nonsingular surface $X$ in $P_k^N$ can be embedded into $P_k^3$ via a projection if and only if

$$\deg(K_X) = (d - 4)d,$$

$$\left( K_X^2 \right) = (d - 4)^2d,$$

$$p_a(X) = \frac{1}{6}(d - 1)(d - 2)(d - 3).$$

Again $d = \deg(X)$, $K_X$ is the canonical divisor and $p_a(X)$ the arithmetic genus of $X$. The necessity of (3) was noted by Iversen [4].

It should be easy to compute formulas similar to (2) and (3) in any dimension $n$ by means of (1), and thus obtain a characterization (in terms of classical invariants like $K_X$, $p_a(X)$) of those nonsingular varieties $X$ in $P_k^N$ which can be projected isomorphically onto a hypersurface in $P_k^{n+1}$. Of course (1) with $m = n+1$ gives such a characterization, in terms of the degrees of certain monomials in the Chern-classes of $X$.

Another application of the theorem is to Abelian varieties. In fact, the question of embeddings for Abelian varieties is resolved as follows: Let $X \subseteq P_k^N$ be an $n$-dimensional Abelian variety. Then:

(i) $X$ can always be embedded into $P_k^{2n+1}$ via a projection from $P_k^N$;

(ii) $X$ can be embedded into $P_k^{2n}$ via a projection from $P_k^N$ $\iff \deg(X) = \frac{1}{2}(2n+1)$;

(iii) $X$ cannot be embedded into $P_k^{2n-1}$.

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*Added in Proof.* Using standard combinatorial identities, one easily checks that

$$(-1)^{j-n}\beta_j = \sum_{i=0}^{i=n} \binom{j-i-1}{j-n-1} d_i - d^2_i.$$
For \( n=1 \), (ii) gives \( \deg(X)=3 \) which is no surprise, and for \( n=2 \) we get \( \deg(X)=10 \). The necessity of this condition for the embedding of a 2-dimensional Abelian variety into \( P_5^4 \) was noted by Horrocks and Mumford in [3, Theorems 5.1 and 5.2].

It should be noted that [2] deals only with embedded projective varieties. For a given projective variety \( X \), one may ask for the least integer \( e=e(X) \) such that \( X \) may be embedded into \( P_5^e \). If \( X \) is given as a subvariety of some \( P_5^N \), one may very well have \( m(X)>e(X) \). Nevertheless, calculation of \( m(X) \) can be used to obtain upper and lower bounds for \( e(X) \), see for example the computation for Abelian varieties referred to above. In order to compute \( e(X) \), one must find the projective embeddings \( i \) of \( X \) for which \( m(X,i) \) is minimal, i.e., for which the embedding obstruction is as nice as possible. We hope to return to this question later.

References