SOME EXAMPLES OF SPHERE BUNDLES OVER SPHERES WHICH ARE LOOP SPACES \( \mod p \)

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ABSTRACT. In this note we give sufficient conditions that certain sphere bundles over spheres, denoted \( B_n(p) \), are of the homotopy type of loop spaces \( \mod p \) for \( p \) an odd prime. The method is to construct a classifying space for the \( p \)-profinite completion of \( B_n(p) \) by collapsing an Eilenberg-Mac Lane space by the action of a certain finite group.

We say that a space \( X \) has some property \( \mod p \) if the localization of \( X \) at \( p \) has the property. The problem of determining which spheres are of the homotopy type of loop spaces \( \mod p \) has been completely solved by Sullivan [9]. It is therefore natural to ask which sphere bundles over spheres are of the homotopy type of loop spaces \( \mod p \). In this regard, results of Curtis [2] and Stasheff [7] concerning the question of which sphere bundles over spheres are \( H \)-spaces \( \mod p \) give some negative information. Moreover, in a recent paper [3] we investigated a certain class of sphere bundles over spheres and gave necessary conditions for them to be of the homotopy type of a loop space \( \mod p \) for \( p \) an odd prime. In this note we prove that certain of these bundles satisfying the conditions of [3] are of the homotopy type of a loop space \( \mod p \) and answer a question posed in [8].

For \( p \) an odd prime and \( n \) a positive integer, the space \( B_n(p) \) is an \( S^{2n+1} \)-bundle over \( S^{2n+1} \) classified by the generator of the \( p \)-primary part of \( \pi_3(S^{2n+1}) \). From [5] we have that \( H^*(B_n(p); \mathbb{Z}/p) \) is an exterior algebra on generators \( x \) and \( y \), where \( \deg x = 2n+1 \), \( \deg y = 2n+2p-1 \) and \( \partial x = y \). Although few of the \( B_n(p) \) are of the homotopy type of a loop space \( \mod p \) (see [3]), we have the following exceptions.

**Theorem 1.** The space \( B_n(p) \) is of the homotopy type of a loop space \( \mod p \) if \( n \) and \( p \) satisfy any of the following conditions:

(i) \( n=1; p=\text{any odd prime} \),
(ii) \( n=p-2; p=\text{any odd prime} \),
(iii) \( n=7; p=17 \),
(iv) \( n=5; p=19 \),
(v) \( n=19; p=41 \).


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REMARK. Two cases of Theorem 1 follow immediately from [5]: namely $B_t(3)$ is of the homotopy type of $\text{Sp}(2) \text{ mod } 3$, and $B_t(5)$ is of the homotopy type of $G_2 \text{ mod } 5$.

In order to prove Theorem 1, we must introduce the $p$-profinite completion of a space as defined in [9]. For precise statements of some of the pertinent theorems, see [4]. If $X$ is a space, let $\hat{X}_p$ denote the $p$-profinite completion of $X$; for notational convenience we make the following conventions:

$$L_n(p) = \text{localization of } B_n(p) \text{ at } p.$$  
$$C_n(p) = \text{$p$-profinite completion of } B_n(p).$$

**THEOREM 2.** The space $C_n(p)$ is of the homotopy type of a loop space if $n$ and $p$ satisfy any one of the conditions of Theorem 1.

**PROOF OF THEOREM 1.** Theorem 1 now follows from Theorem 2 using techniques of [9]. Suppose $C_n(p)$ is a loop space, and let $BC_n(p)$ denote the classifying space. Let $W$ denote the homotopy pull-back in the following diagram:

$$\begin{array}{ccc} 
W & \rightarrow & BC_n(p) \\
\downarrow & & \downarrow \\
K(Q, 2n + 2) \times K(Q, 2n + 2p) & \rightarrow & K(Q_p, 2n + 2) \times K(Q_p, 2n + 2p),
\end{array}$$

where $Q_p$ denotes the $p$-adic numbers. Looping the diagram we conclude that $L_n(p) \simeq \Omega W$. Q.E.D.

The proof of Theorem 2 is somewhat involved and so we outline the procedure. Given $n$ and $p$ satisfying one of the conditions, we construct two $p$-profinately complete spaces, $A$ and $X$, together with a map $i: A \rightarrow X$. We show $\Omega A \simeq S_{2n+1}^{\mathbb{Z}_p}$, $H^*(\Omega X; Z/p) \simeq H^*(B_n(p); Z/p)$ as modules over the Steenrod algebra, and $(\Omega i)^*: H^{2n+1}(\Omega X; Z/p) \rightarrow H^{2n+1}(\Omega A; Z/p)$ is an isomorphism. We conclude that there is a map $f: S_{2n+1} \rightarrow \Omega X$ such that $f^*: H^{2n+1}(\Omega X; Z/p) \rightarrow H^{2n+1}(S_{2n+1}; Z/p)$ is an isomorphism. From [5] we have the following cell structure for $B_n(p)$:

$$B_n(p) \cong S_{2n+1} \cup_a e^{2n+2p-1} \cup e^{4n+2p}.$$

Since $\mathcal{P}$ is nontrivial on $H^*(\Omega X; Z/p)$, we conclude $fX$ is null homotopic. Therefore, by proving $\pi_{4n+2p-1}(\Omega X)$ is trivial, we have shown that $f$ extends to a map $f: B_n(p) \rightarrow \Omega X$. By functoriality of $\mathcal{P}$ and cup products, this extension induces an isomorphism of mod $p$ cohomology. From [9] or [4] we have that $C_n(p) \simeq \Omega X$.

In this note we give details of the construction only in the case $n=1$. The remaining cases are similar and details can be found in [4]. Let $\mathbb{Z}_p$ denote the $p$-adic integers, and let $\theta$ be a primitive $(p+1)$st root of unity.
It is easily verified that $\theta + \theta^{-1}$ and $(\theta - \theta^{-1})^2$ are in $\mathcal{Z}_p$. Let $D_{p+1}$ denote the dihedral group of order $2(p+1)$ in $GL(2, \mathcal{Z}_p)$ generated by

$$\frac{1}{2} \begin{pmatrix} \theta + \theta^{-1} & (\theta - \theta^{-1})^2 \\ 1 & \theta + \theta^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Let $C_2$ denote the cyclic group of order 2 in $D_{p+1}$ generated by the second element above. Let $j : \mathcal{Z}_p \to \mathcal{Z}_p \times \mathcal{Z}_p$ be inclusion into the first factor. We proceed as in [1]. The natural actions of $C_2$ and $D_{p+1}$ on $\mathcal{Z}_p$ and $\mathcal{Z}_p \times \mathcal{Z}_p$ induce actions on the Eilenberg-Mac Lane spaces $K(\mathcal{Z}_p, 2)$ and $K(\mathcal{Z}_p \times \mathcal{Z}_p, 2)$. Let $ED_{p+1}$ be an acyclic complex on which $D_{p+1}$ acts freely, and let $C_2$ and $D_{p+1}$ act on $K(\mathcal{Z}_p, 2) \times ED_{p+1}$ and $K(\mathcal{Z}_p \times \mathcal{Z}_p, 2) \times ED_{p+1}$ by diagonal actions. Let $A$ and $X$ denote the $p$-profinite completions of the respective orbit spaces. Denote by $i : A \to X$ the map induced by $j$. From [1] we can conclude:

$$H^*(A; \mathbb{Z}/p) \approx \mathbb{Z}/p[x], \quad \deg x = 4;$$
$$H^*(X; \mathbb{Z}/p) \approx \mathbb{Z}/p[u,v], \quad \deg u = 4, \quad \deg v = 2p + 2;$$
$$i^*(u) = x \quad \text{and} \quad \mathcal{P}^i u = v \quad \text{(see [2]).}$$

If we consider loop spaces we have correspondingly:

$$H^*(\Omega A; \mathbb{Z}/p) \approx E(\bar{x}), \quad \deg \bar{x} = 3;$$
$$H^*(\Omega X; \mathbb{Z}/p) \approx E(\bar{u}, \bar{v}), \quad \deg \bar{u} = 3, \quad \deg \bar{v} = 2p + 1;$$
$$(\Omega i)^*(\bar{u}) = \bar{v} \quad \text{and} \quad \mathcal{P}^i \bar{u} = \bar{v}. $$

Since $\Omega A$ is a simply-connected, $p$-profinitley complete space, we have $\Omega A \simeq S^3_p$. Therefore we get a map $f : S^3 \to \Omega X$ such that $f^*(\bar{u}) \neq 0$. From the above remarks, to extend $f$ to $B_1(p)$ we need only show $\pi_{2p+3}(\Omega X) = 0$.

Consider the diagram:

$$\begin{array}{ccc}
\Omega X & \longrightarrow & \Omega X \\
\downarrow & & \downarrow \\
E & \longrightarrow & \Lambda X \\
\downarrow & & \downarrow \\
A & \longrightarrow & X
\end{array}$$

The pull-back $E$ is a simply-connected $p$-profinitlely complete space. Moreover, we can compute $H^*(E; \mathbb{Z}/p)$ from the Eilenberg-Moore spectral sequence, which collapses [6] and gives $H^*(E; \mathbb{Z}/p)$ as an exterior algebra on a generator of degree $2p+1$. We conclude $E \simeq S^2_{2p+1}$. Since $\pi_{2p+4}(A) \approx \pi_{2p+3}(S^3_p) = 0$ and $\pi_{2p+3}(S^2_{2p+1}) = 0$, we have $\pi_{2p+3}(\Omega X) = 0$.

Therefore, $f$ extends to a map $f : B_1(p) \to \Omega X$ which induces an isomorphism on mod $p$ cohomology. We conclude that $C_1(p) \simeq \Omega X$. 


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