ZEROS OF DERIVATIVE OF RIEMANN'S ξ-FUNCTION

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Riemann's ξ-function is defined by 

ξ(s)=H(s)ζ(s)

where ζ(s) is the Riemann zeta-function and 

H(s)=\frac{1}{2}(s^2-s)\pi^{-s/2}\Gamma(s/2).

The functional equation is ξ(s)=ξ(1-s). Moreover ξ is an entire function which has as its zeros precisely those of ζ in the critical strip. Because ξ(\frac{1}{2}+it) is real, it follows that between consecutive zeros of ξ on the \frac{1}{2}-line there is at least one zero of ξ'.

It has recently been shown [1], [2] that ζ(s) has at least \frac{1}{3} of its zeros in the critical strip on σ=\frac{1}{2}. Here a similar result will be proved for ξ'(s) (for which \frac{1}{3} is already implied by the remark above). Let U=T/log^{10} T. Then the following theorem will be sketched.

THEOREM. More than \frac{1}{10} of the zeros of ξ'(s) in T<t<T+U occur on σ=\frac{1}{2}.

By Stirling's formula, for |σ|<10, H(s)=e^{F(s)}, where

F'(s)=\frac{1}{2}\log s/2\pi + O(1/s),

F''(s)=O(1/s).

From ξ(s)=H(s)ξ(s)=H(1-s)ξ(1-s) follows

ξ'(s)=H'(s)ξ(s) + H(s)ξ'(s)

(1)

=-H'(1-s)ξ(1-s) - H(1-s)ξ'(1-s).

and also

H''(s)ζ(s) + 2H'(s)ζ'(s) + H(s)ζ''(s)

= H''(1-s)ζ'(1-s) + 2H'(1-s)ζ'(1-s) + H(1-s)ζ''(1-s).

Since H'=HF', and H''=HF'+HF'',

F'(s)[H'(s)ζ(s) + H(s)ζ'(s)]

-F'(1-s)[H'(1-s)ζ(1-s) + H(1-s)ζ'(1-s)]

= -H'(s)ζ'(s) - H(s)ζ''(s) - H(s)F''(s)ζ(s)

+ H'(1-s)ζ''(1-s) + H(1-s)ζ''(1-s)

+ H(1-s)F''(1-s)ζ(1-s).
By (1) this can be written as

\[(F'(s) + F'(1 - s))\xi'(s)\]

\[(= - H(s)[F'(s)\zeta'(s) + \zeta''(s) + F''(s)\xi(s)] + H(1 - s)[F'(1 - s)\zeta'(1 - s) + \zeta''(1 - s) + F''(1 - s)\xi(1 - s)].\]

From the functional equation, \(-2\xi'(s) = -\xi'(s) + \xi'(1 - s),\) and so

\[-2\xi'(s)(F'(s) + F'(1 - s))
= -(F'(s) + F'(1 - s))(H'(s)\zeta(s) + H(s)\xi'(s))
+ (F'(s) + F'(1 - s))(H'(1 - s)\zeta(1 - s) + H(1 - s)\xi'(1 - s)).\]

Adding the above to (2) gives

\[-2\xi'(s)(F'(s) + F'(1 - s))
= -(F'(s) + F'(1 - s))(H'(s)\zeta(s) + H(s)\xi'(s))
+ (F'(s) + F'(1 - s))(H'(1 - s)\zeta(1 - s) + H(1 - s)\xi'(1 - s)).\]

Let

\[G(s) = \zeta(s) + \zeta'(s)/F'(s)
+ [F'(s) + F'(1 - s)]^{-1}(\zeta'(s) + \zeta''(s)/F'(s) + F''(s)\xi(s)/F'(s)).\]

Then (3) becomes

\[\xi'(s) = F'(s)H(s)G(s) - F'(1 - s)H(1 - s)G(1 - s).\]

For \(s = \frac{1}{2} + it,\) the right side above is the difference between two complex conjugate quantities. Hence \(\xi'(\frac{1}{2} + it) = 0,\) where

\[\text{arg}(F'HG(\frac{1}{2} + it)) \equiv 0 \pmod{\pi}.\]

Since \(F' \sim (\log t/2\pi)/2,\) it has little effect on the change in argument as \(t\) increases. By Stirling's formula \(\text{arg} H(\frac{1}{2} + it)\) changes rapidly and by itself would supply the full quota of zeros of \(\xi'(s)\) on \(\sigma = \frac{1}{2}.\) However \(G\) also plays a role. What will be shown is that the change in \(\text{arg} G\) is sufficiently restricted so that it cancels less than 30\% of the change in \(\text{arg} H.\)

To get the change in \(\text{arg} G(\frac{1}{2} + it),\) the principle of the argument can be used. The determination of the number of zeros of \(G(s)\) in a rectangle \(D\) with vertices \((\frac{1}{2} + iT, 3 + iT, \frac{1}{2} + i(T + U), 3 + i(t + U))\) leads in a familiar way to the change in \(\text{arg} G\) on \(\sigma = \frac{1}{2}, T < t < T + U.\) To get the number of zeros of \(G\) in \(D,\) Littlewood's lemma [3, §9.9] is used in a familiar way [3, §9.15]. However it turns out to be more efficient to first multiply \(G\) by an
entire function \( \psi(s) \) even though this may introduce extra zeros. The key term in the estimate of the number of zeros of \( \psi G \) in \( D \) is

\[
\int_T^{T+U} \log |\psi G(a + it)| \, dt/2\pi(\frac{1}{2} - a),
\]

where \( a < \frac{1}{2} \) and \( \frac{1}{2} - a \) is small. Use is now made of

\[
\int_T^{T+U} \log |\psi G(a + it)| \, dt \leq \frac{U}{2} \log \left( \frac{1}{U} \int_T^{T+U} |\psi G(a + it)|^2 \, dt \right).
\]

The choice for \( \psi \) is

\[
\psi(s) = \sum_{j \leq \psi} \frac{\mu(j) \log j}{j^{1/2-a} \log j},
\]

and \( \psi = T^{1/2}/\log^2 T \). To compute

\[
J = \frac{1}{U} \int_T^{T+U} |\psi G(a + it)|^2 \, dt
\]

it is necessary to express \( \zeta \) and its derivatives in terms of the approximate functional equation as done in [1], [2]. Let \( R = (\frac{1}{2} - a) \log T/2\pi \). Then lengthy calculations lead to

\[
J = e^{2R} \left( \frac{1}{24R} - \frac{1}{12R^2} + \frac{2}{3R^3} - \frac{3}{R^4} + \frac{6}{R^6} \right)
\]

\[
- \frac{R}{12} + \frac{3}{4} - \frac{29}{24R} - \frac{13}{4R^2} - \frac{20}{3R^3} - \frac{9}{R^4} - \frac{6}{R^5}.
\]

\[
+ O\left( \frac{(\log \log T)^{16}}{\log T} \right).
\]

For \( R = 1.1 \), \( J \leq 1.3634 \). Therefore

\[
\frac{1}{\frac{1}{2} - a} \int_T^{T+U} \log |\psi G(a + it)| \, dt \leq \frac{U}{2(\frac{1}{2} - a)} \log 1.3634 = U \log T/2\pi \frac{\log 1.3634}{2R} \leq 0.1414 U \log T/2\pi.
\]

with \( R = 1.1 \). Thus the change in arg \( G \) is at most 0.1414 \( U \log T/2\pi \). By Stirling's formula the change in arg \( H(\frac{1}{2} + it) \) is essentially \( \frac{1}{2} U \log T/2\pi \), and so the change in arg \( (HF'G) \) is at least 0.3586 \( U \log T/2\pi \). Since the zeros occur mod \( \pi \), the number is at least 0.7172 \( U(\log T/2\pi)/2\pi \) which is more than 0.7 of the total number in \( T < t < T + U \).
REFERENCES


2. N. Levinson, *More than one third of zeros of Riemann's zeta-function are on \( \sigma = \frac{1}{2} \)*, Advances in Math. (to appear).


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