ON THE EXTENSION OF BOUNDARY INTEGRABLE ALMOST COMPLEX STRUCTURE

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1. Introduction. Let \( \{M, M'\} \) be a finite Kähler manifold, i.e., \( M' \) is a complex Kähler manifold, \( M \) is an open submanifold of \( M' \) with compact closure, \( M_0 = bM \), the boundary of \( M \), is a \( C^\infty \) submanifold of \( M' \), and for each \( p \in M_0 \) there exists a coordinate neighborhood \( U \) of \( p \) with real coordinates \( t^1, \ldots, t^{2n-1}, r \) such that \( r(q) < 0 \) for \( q \in U \cap M \) and \( r(q) > 0 \) for \( q \in U \cap (M' - M) \). It is assumed that the following conditions hold:

A. For each boundary point the Levi form has at least two positive eigenvalues.

B. There exists a constant \( c_0 > 0 \) such that for all \( u \in C^0(M_0) \), \( q=1, 2 \) \((2\square - \triangle) u, u \geq c_0 (u, u)\) where \( \Theta \) is the holomorphic tangent bundle of \( M' \), \( C^{0,q}(M, \Theta) \) is the space of all \( C^0 \) \( \Theta \)-valued \((p, q)\)-forms extendible to a neighborhood of \( M \), \( \square \) (resp., \( \triangle \)) is the complex (resp., the real) Laplacian on \( C^{p,q}(M, \Theta) \) and \((, )\) is the \( L^2 \)-inner product over \( M \) (see [2]).

Then the main result of this note states that a sufficiently small integrable almost-complex structure on \( M_0 \) can be extended to a complex structure on \( M \). A complete proof will appear elsewhere; a brief outline follows.

However, we first take a closer look at condition B. Let \( D \) be the covariant differentiation operator associated with the connection \( \theta \) of the metric \( g \) on \( M' \), i.e.,

\[
Du = d\theta + \theta \wedge u = \delta u + \delta u
\]

for \( u \in C^{p,q}(M, \Theta) \). Let \( D^* \) and \( \delta^* \) be the formal adjoints of \( D \) and \( \delta \), respectively. Then \( \triangle = DD^* + D^*D \) and \( \square = \delta \delta^* + \delta^* \delta \). Since \( g \) is Kähler, \( \triangle = 2\square - K \), \( K = \sqrt{-1} e(\delta \Lambda - \Lambda e(\delta)) \), where

\[
e(\delta) u = \delta \theta \wedge u, \quad \Lambda u = \star^{-1}(\rho \wedge \star u),
\]

\( \star \) is the Hodge star operator and \( \rho \) is the Kähler form of \( g \). We refer


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to [3, pp. 482-483], for verification of this identity. Hence, condition B requires the existence of a constant \( c_0 > 0 \) such that \( (Ku, u) \geq c_0(u, u) \) for all \( u \in C^{0,q}(\mathcal{M}, \Theta) \), \( q = 1, 2 \). Now it is established in [2, p. 276], that if the scalar curvature is sufficiently negative, then one has the stronger result \( (Ku, u) \geq c_0(u, u) \) for all \( x \in M' \), where \( \langle , \rangle_x \) is the inner product at the point \( x \), i.e., \( \Theta \) is \( W^{0,q} \)-elliptic. It is also shown in [2] that the criterion of \( W \)-ellipticity is satisfied for a large class of bounded homogeneous domains in \( C^n \) provided with the Bergman metric. More generally, let \( M' \) be a manifold whose universal covering space \( M' \) is isomorphic to \( D_1 \times \cdots \times D_r \), where \( D_i \) is a bounded irreducible symmetric domain with \( \dim C D_i \geq 3 \). Then \( \Theta \) is \( W^{0,q} \)-elliptic for \( 0 \leq q \leq 2 \), and condition B will hold for any relatively compact open submanifold \( M' \) of \( M \) with smooth boundary.

2. Definitions and notation. Let \( M_0 \) be a \( C^\infty \) manifold of real dimension \( 2n-1 \) and let \( CTM_0 \) be the complexified tangent bundle.

2.1. Definition. An almost-complex structure on \( M_0 \) is given by a complex subbundle \( E'' \) of \( CTM_0 \) of fiber complex dimension \( n-1 \) such that \( E'' \cap E'' = \{0\} \).

2.2. Definition. The almost-complex structure \( E'' \) on \( M_0 \) is integrable if, for any two sections \( L \) and \( L' \) of \( E'' \) over an open set \( U \) of \( M_0 \), \([L, L']\) is also a section of \( E'' \).

We now assume that \( M_0 \) is the boundary of a finite complex manifold \( \{M, M'\} \). The complex structure on \( M' \) induces an integrable almost-complex structure \( T'' \) on \( M_0 \).

2.3. Definition. The almost-complex structure \( E'' \) on \( M_0 \) is of finite distance from \( T'' \) if \( \pi''|E'':E'' \to T'' \) is an isomorphism where \( \pi'':CTM_0 \to T'' \) is the projection.

In this case \( E'' = \{X-\tau \circ \varphi(X)|X \in T''\} \) where \( \tau: \Theta|M_0 \to T' \oplus CF \) is an isomorphism, \( T'' = T' \), \( CF \) is the complexification of a real one-dimensional subbundle \( F \) of \( TM_0 \) such that \( CTM_0 = T'' \oplus T'' \oplus CF \) and

\[
\varphi = -\pi^{-1} \circ (id - \pi'') \circ (\pi''E'')^{-1}: T'' \to \Theta \big| M_0,
\]

i.e., \( \varphi \) is a \( \Theta|M_0 \)-valued \( C^\infty \) differential form on \( M_0 \) of type \((0, 1)_b\). Conversely, any such differential form \( \varphi \) gives rise to an almost complex structure \( E'' \) on \( M_0 \). We will denote \( E'' \) by \( T'_{\varphi} \). As in the case of complex manifolds, there exists a \( \Theta|M_0 \)-valued \( C^\infty \) differential form \( \Phi \) on \( M_0 \) of type \((0, 2)_b \) such that \( \Phi = 0 \) if and only if \( T'_{\varphi} \) is integrable.

Let \( \varphi \) be a \( T' \)-valued form and let \( \omega \in C^{0,1}(\mathcal{M}, \Theta) \) be such that \( t \omega \), the complex tangential part of \( \omega \), is equal to \( \varphi \) on \( M_0 \). Let \( \Omega = \delta \omega - [\omega, \omega] \). If \( T''_{\omega} \) is the almost complex structure on \( M \) induced by \( \omega \), then one can show that \( T''_{\omega} = CTM_0 \cap T''_{\omega} \) and \( t \Omega = 0 \) on \( M_0 \) if and only if \( \Phi = 0 \).
3. The main result. Now we can formulate the following extension problem.

THEOREM. Let \( \{M, M'\} \) be a finite complex Kähler manifold such that conditions A and B in §1 are satisfied. Let \( \varphi \) be a \( T' \)-valued \( C^\infty \) differential form of type \((0, 1)_b \) with sufficiently small Hölder norm \( |\varphi|_{k+\alpha}, 0<\alpha<1 \), for some integer \( k>0 \) depending on \( n \). Assume that \( T'_\varphi \) is integrable. Then there exists \( \omega \in C^{0,1}(\overline{M}, \Theta) \) such that \( \Omega=0 \) and \( \omega=T'_\varphi \) on \( M_0 \).

We first consider the quadratic form

\[
Q(u, v) = \frac{1}{2}[(D_u, D_v) + (D^*u, D^*v) + (K_u, v)] - 2([\psi, u], \delta v)
\]

for some \( \psi \in C^{0,1}(\overline{M}, \Theta) \) with sufficiently small norm and \( u, v \in \mathcal{B} = \{\omega \in C^{0,1}(\overline{M}, \Theta)|\omega_0=0 \text{ on } M_0\} \). One can easily check that by condition B, \( \text{const} N^2(u) \leq |\text{Re } Q(u, u)| \leq \text{const} N^2(u) \) where \( \text{Re} \) stands for the real part of \( Q(u, u) \), and \( N^2(u)=\|u\|^2+\|Du\|^2+\|D^*u\|^2 \). Hence, if \( \|u\|_s \) is the Sobolev \( s \)-norm of \( u \), then \( \|u\|_1 \leq \text{const} |\text{Re } Q(u, u)| \).

It follows from the theory developed in [1] and [4] that for each \( \sigma \in C^{0,1}(\overline{M}, \Theta) \) there exists a unique \( u \in \mathcal{B} \) such that \( Q(u, v)=(\sigma, v) \) for all \( v \in \mathcal{B}_\Sigma \), the completion of \( \mathcal{B} \) with respect to the norm \( N \) such that

\[
\begin{align*}
(1) & \quad \|u\|_{s+2} \leq c_s \|\sigma\|_s; \\
(2) & \quad L_\psi u = \frac{1}{2}(DD^* + D^*D + K)u - 2\delta^*[\psi, u] = \sigma; \\
(3) & \quad tD^*u = 0 \quad \text{on } M_0; \\
(4) & \quad |u|_{k+\alpha+2} \leq c_k \|\psi\|_{k+\alpha}
\end{align*}
\]

for sufficiently large \( k \). The constants \( c_s \) and \( c_k \) depend on \( s \) and \( k \) and on the derivatives of \( \psi \) up to order \( s \) and \( k \), respectively. If \( |\psi|_{k+\alpha} \) is sufficiently small we may assume that \( c_k \) in (4) depends only on \( k \).

We observe that \( D^*u=-\Delta^* u=\Delta^* u \), and since \( u \) is a form of type \((0, 1)_b \), \( \Delta^* u=0 \) and \( D^*u=\delta^* u \). On the other hand for a Kähler metric \( g \) the complex Laplacian \( \Delta=\delta^* \delta + \delta \Delta \) is \( \frac{1}{2}(DD^* + D^*D + K) \), and if \( \sigma=\delta^* h \) for \( h \in C^{0,2}(\overline{M}, \Theta) \), then (3) and Stokes' theorem imply that

\[
L_\psi u = \sigma \text{ if and only if } \Delta\delta u = 2\delta^*[\psi, u] = \delta^* h.
\]

We now consider the nonlinear differential system \( \delta^* \Omega=0 \). Let \( \omega_0 \in C^{0,1}(\overline{M}, \Theta) \) be an extension of \( \varphi \) such that \( |\omega_0|_{k+\alpha} \leq \text{const} |\varphi|_{k+\alpha} \). One can inductively construct a sequence of approximate solutions \( \omega_{m+1}=\omega_m+u_m \), where \( u_m \) is the solution of (5) with \( tu_m=t\delta^* u_m=0 \) on \( M_0 \), \( \psi=\omega_m \), \( h=-\Omega_m=-\delta \omega_m+[\omega_m, \omega_m] \). Since \( |\delta^* \Omega_m|_{k+\alpha} \leq \text{const} |u_m|_{k+\alpha} \), (4) implies that there exists a constant \( c>0 \) such that

\[
|\omega_{m+1} - \omega_m|_{k+\alpha} \leq c |\omega_m - \omega_{m-1}|_{k+\alpha}.
\]
for $m=1, 2, \cdots$. This is enough to conclude that there exists a $\Theta$-valued form $\omega$ of type $(0, 1)$ and of class $C^{k+a}$ on $M$ such that $\bar{\partial}\omega = \varphi$ on $M_0$, and $|\omega|_{k+a} \leq \text{const} |\varphi|_{k+a}$.

Now it can easily be shown that $\bar{\partial}\Omega = 2[\omega, \Omega]$. By condition A and the fact that the normal part of $\bar{\partial}\Omega$ vanishes on $M_0$, the basic estimate of the $\bar{\partial}$-Neumann problem holds for $\bar{\partial}\Omega$, i.e.,

$$E(\bar{\partial}\Omega) \leq \text{const}(\|\Omega\|^2 + \|\bar{\partial}\Omega\|^2 + \|\bar{\partial}\omega\|^2).$$

For the definition of the norm $E$, we refer to [5] and [6]; the operators $*$ and $\#$ are defined in [2]. Then by condition B and the complete continuity of $E$, one can obtain the estimate $\|\bar{\partial}\Omega\| \leq c_0|\varphi|_{k+a} \|\bar{\partial}\omega\|$ for some constant $c_0$. Thus $\bar{\partial}\Omega = 0$ if $|\varphi|_{k+a}$ is sufficiently small. Since $t\Omega = 0$ on $M_0$, $\bar{\partial}\Omega = 0$, and $\bar{\partial}\omega = 0$, condition B implies that $\Omega = 0$.

Finally, it follows from the construction of approximate solutions that $\omega = \omega_0 + w$, where $w$ is of class $C^{k+a}$ and $\bar{\partial}w = 0$. Then $\bar{\partial}(\bar{\partial}\omega - [\omega, \omega]) = 0$ can be expressed as

$$\Box w - \bar{\partial}((2[\omega_0, w] + [w, w]) = \bar{\partial}((\omega_0, \omega_0) - \omega_0).$$

This equation is elliptic if $|\varphi|_{k+a}$ is sufficiently small. Since $\omega_0$ is of class $C^\infty$, $w$ is also of class $C^\infty$.

REFERENCES