RADICAL EMBEDDING, GENUS, AND TOROIDAL DERIVATIONS OF NILPOTENT ASSOCIATIVE ALGEBRAS

BY F. J. FLANIGAN

Communicated by Mary Gray, March 11, 1974

ABSTRACT. The author continues to discuss this problem: given a nonzero nilpotent finite-dimensional associative algebra \( N \) over the perfect field \( k \), describe the set of unital associative \( k \)-algebras \( A \) satisfying the equation \( \text{rad} A = N \), together with the "nowhere triviality" condition \( \text{Ann}_A N < N \). In this paper the Lie homomorphism \( \delta : \text{SLie} \rightarrow \text{Der}_k N \) induced by bracketing (where \( A \) has Wedderburn decomposition as semidirect sum \( S + N \)) is studied as follows: (i) the kernel and image of \( \delta \) are computed; (ii) conditioning the derivation algebra \( \text{Der}_k N \) conditions the semisimple \( S \); (iii) for instance, \( \text{Der}_k N \) solvable implies that \( S \) is a direct sum of fields; (iv) those tori in \( \text{Der}_k N \) of the form \( OS \) are characterized in terms of their 0-weightspace in \( N \).

1. Introduction. For previous discussions, see Hall [2] and Flanigan [1]. Throughout, \( N \) is a given finite-dimensional nilpotent \( k \)-algebra with \( k \) perfect. We seek those semisimple \( k \)-algebras \( S \) which satisfy the following conditions.

(1.1) DEFINITION [1]. \( N \) accepts \( S \) as a nowhere trivial Wedderburn factor if there is a unital associative \( k \)-algebra \( SA \) such that (i) \( A \cong N + S \) (Wedderburn decomposition), and (ii) \( S \cap \text{Ann}_A N = (0) \).

Note that (ii) forces \( A \) to be finite dimensional, and that \( N \neq (0) \) implies \( S \neq (0) \). In [1] we examined candidates \( S \) for acceptance by considering such invariants of \( N \) as its quotients \( N/N^i \) and its graded form \( \text{gr} N \). Now we utilize the Lie algebra \( \text{Der}_k N \) of \( k \)-algebra derivations \( N \rightarrow N \) by noting that, if \( N \) accepts \( S \) as in (1.1), then there is a Lie homomorphism

\[
\delta : \text{SLie} \rightarrow \text{Der}_k N
\]

with \( \delta(b)x = [b, x] = bx - xb \) for all \( x \) in \( N \), \( b \) in \( S \), and with the products taken in \( A \).

We are particularly interested in those \( S \) which are direct sums of fields. Reason: the center of every semisimple algebra accepted by \( N \) would be of this type. These direct sums of fields are determined by the
"genus" of $N$ (§3). Thus, genus($N$) = 0 means that the only $S$, commutative or not, accepted by $N$ is essentially that obtained by the well-known process of adjoining a unity to $N$. In §4 we bound genus($N$) in terms of the dimension of maximal tori in Der$_k N$ and from this draw consequences for $S$. The family of examples in §5 shows that this upper bound on genus($N$) may or may not be attained for a given $N$, and if not, it is because there exists an abelian $S_{\text{Lie}}$ and Lie homomorphism $S_{\text{Lie}} \rightarrow$ Der$_k N$ which is not induced by bracketing (see (1.2)) in an associative $A = N + S$. Finally, we identify those tori ("Peirce tori") in Der$_k N$ which are of the form $\delta(S_{\text{Lie}})$ in terms of the associative algebra structure of their 0-weightspaces in $N$ (§6). The Peirce tori are those whose weight-space decomposition of $N$ is essentially a Peirce (idempotent) decomposition in the classical sense.

It is a pleasure to acknowledge helpful conversations and correspondence with Robert Kruse, George Leger, and James Malley.

2. The Lie homomorphism $\delta$. It often makes good sense to specify that (i) $N$ is indecomposable (into two-sided ideals) as a $k$-algebra, and that (ii) the semisimple $k$-algebra $S$ is split over $k$, that is, $S$ is an ideal direct sum of total matrix algebras $M(r_a, k)$ of rank $r_a$ with all entries in $k$. This latter is always the case if $k$ is algebraically closed.

(2.1)Lemma. Let the nonzero indecomposable nilpotent $k$-algebra $N$ accept the split semisimple $S = \bigoplus_a M(r_a, k)$, with $a = 1, \cdots, s$, as a nowhere trivial Wedderburn factor. Then the map $\delta$ of (1.2) satisfies

(i) ker($\delta$) is the one-dimensional Lie ideal $k\cdot 1 = k\cdot 1_S$;
(ii) if char($k$) divides none of the ranks $r_a$, then the image of $\delta$ is isomorphic with the Lie algebra

\[
\left( \bigoplus_a \text{sl}(r_a, k) \right) \oplus \left( \bigoplus_a k\cdot e_a \right) / k\cdot 1
\]

where $e_a$ is the unity element of $M(r_a, k)$;
(iii) $S$ is a direct sum of copies of $k$, that is, all $r_a = 1$ if and only if the image of $\delta$ is a torus in Der$_k N$.

(2.2) Comments. (a) If $s \geq 2$, then ker($\delta$) is therefore a proper subalgebra of the center of $S_{\text{Lie}}$;
(b) The Lie algebra $\text{sl}(r_a, k)$ consists of the matrices with trace zero;
(c) $k\cdot e_a$ is the subalgebra of scalar matrices in $M(r_a, k)$, and $1 = 1_S = e_1 + \cdots + e_s$;
(d) A torus is an abelian linear Lie algebra consisting of semisimple operators;
(e) The Lemma follows from elementary considerations of two-sided matrix actions on $N$. The proof does not require nilpotence of $N$,
but only that \( N \) be an ideal in \( A=S+N \). The Lemma is false, however, if \( N \) is itself decomposable as a direct sum of two-sided ideals.

(2.3) **Question.** If \( N \) accepts a maximal (see [1]) split \( S \), does the map \( \delta \) always send the center of \( S \) into the solvable radical of \( \text{Der}_k N \)? An affirmative answer would yield a much more severe constraint on \( S \). This would be reflected in statement (iii) of Theorem (4.1), where the integer \( t(N) \) could then be replaced by a smaller and better understood number, the dimension of a maximal torus in the solvable radical of \( \text{Der}_k N \).

3. **Genus(\( N \)) and \( t(\mathcal{N}) \).** The genus will provide a measure of the “fineness” of the Peirce decompositions which \( N \) admits.

(3.1) **Definition.** If \( N \) is a nonzero nilpotent \( k \)-algebra, then \( \text{genus}(N)=\max_x (\dim_k S_x)-1 \), where \( S=\bigoplus_1^x s \) is a direct sum of \( s \) copies of the field \( k \) accepted by \( N \) as a nowhere trivial Wedderburn factor.

Thus \( \text{genus}(N) \geq 0 \) and, if \( N=I_1 \oplus \cdots \oplus I_q \) is a decomposition into nonzero two-sided ideals, then one readily checks that \( \text{genus}(N)=q-1+\sum \text{genus}(I_i) \), and that this is \( \geq 1 \) if \( q \geq 2 \).

(3.2) **Example.** Let \( N \) be the nilpotent algebra of all strictly upper triangular \( n \) by \( n \) matrices over \( k \). Then \( \text{genus}(N)=n-1 \). See [1, (2.3)].

(3.3) **Example.** Let \( N \) be the truncated polynomial ideal generated by linearly independent (over \( k \)) noncommuting elements \( x_1, \cdots, x_m \) such that every monomial of degree \( \geq v+1 \) reduces to zero, so that \( N^+ \neq (0) \) but \( N^{v+1}=(0) \). If \( v \geq 2 \), then \( N \) is indecomposable and \( \text{genus}(N)=0 \) independent of \( m \). See [1, (2.4)].

The following invariant of \( N \) was introduced by Leger and Luks [3, §1] to study nilpotent Lie algebras.

(3.4) **Definition.** \( t(N) \) is the dimension of a maximal torus in the derivation algebra \( \text{Der}_k N \).

Thus, if \( \text{Der}_k N \) is nilpotent, then \( t(N)=0 \).

4. **Results on \( S \).** These follow from Lemma (2.1) and the basic structure of algebraic Lie algebras.

(4.1) **Theorem.** Let the nonzero indecomposable nilpotent \( k \)-algebra \( N \) accept as nowhere trivial Wedderburn factor the split semisimple \( S=\bigoplus \alpha \text{M}(r_\alpha, k) \), with \( \alpha=1, \cdots, s \). Then

(i) if \( \text{char } k=0 \) and a Levi factor of \( \text{Der}_k N \) has no nonzero subalgebras \( s\mathfrak{l}(n, k) \), then \( S \) is a direct sum of copies of \( k \);

(ii) if \( \text{Der}_k N \) is solvable, then \( S \) is a direct sum of copies of \( k \);

(iii) \((\sum \alpha r_\alpha)-1 \leq \text{genus}(N) \leq t(N) \);

(iv) in particular, if \( \text{Der}_k N \) is nilpotent, then \( \text{genus}(N)=0 \), that is, \( S=k \).
5. An illustration. This family of algebras will provide counterexamples to the converses of certain assertions in (2.1) and (4.1). Let char $k \neq 2$ and, for each $\tau$ in $k$, let $N_\tau$ be the 3-dimensional $k$-algebra with basis $x, y, z$ and multiplication $xy = z, yx = \tau z$, and all other products of basis elements zero. Note that $(N_\tau)^6 = (0)$, so that $N_\tau$ is associative, nilpotent, and indecomposable.

The following assertions about $N_\tau$ are easily verified.

(5.1) $N_0$ accepts $S = k e_1 \oplus k e_2 \oplus k e_3$ (cf. 3 by 3 upper triangular matrices). Genus$(N_0) = 2$. Also $t(N_0) = 2$.

(5.2) For $\tau \neq 0$, genus$(N_\tau) = 0$, but again $t(N_\tau) = 2$.

(5.3) All Der$_k N_\tau$ are solvable nonnilpotent with 2-dimensional maximal torus.

(5.4) The maximal tori in all Der$_k N_\tau$ are isomorphic, and all modules $N_\tau$ are equivalent.

(5.5) Moral. The structure of Der$_k N$ and its natural representation on $N$ (as discussed so far) are not sufficient to decide genus$(N)$. The conditions we give in §6 that a torus in Der$_k N$ be of the form $\delta(S)$ as in (1.2) must necessarily involve the associative product in $N$.

6. Peirce tori and direct sums of fields. We characterize in terms of Der$_k N$ the direct sums of fields accepted by $N$. Here "eigenvalues" and "weights" refer to the natural representation of Der$_k N$ on $N$.

(6.1) Definition. Let $N$ be an indecomposable nilpotent $k$-algebra. The torus $T$ in Der$_k N$ is a Peirce torus if either $T = (0)$ or $T$ has a spanning set $e_1, \ldots, e_m$ with $m \geq 2$ satisfying these four conditions:

(a) $e_1 + \cdots + e_m = 0$, but any $m-1$ of the $e_i$ furnish a $k$-basis for $T$;
(b) the set of eigenvalues for each $e_i$ is either $\{0, 1\}$, $\{0, -1\}$, or $\{0, 1, -1\}$;
(c) each nonzero weight of $T$ is of the form $\lambda_{ij}$, defined by $\lambda_{ij}(e_i) = 1, \lambda_{ij}(e_j) = -1, \lambda_{ij}(e_k) = 0$ for $h, i, j$ distinct;
(d) the 0-weightspace $W_0$ in $N$ decomposes as $k$-algebra into a direct sum $\bigoplus_i W_i$ of two-sided (possibly zero) ideals, $i = 1, \cdots, m$, satisfying (here $W_{ij}$ is the $\lambda_{ij}$-weightspace) for distinct $h, i, j$,

$$W_i W_j \subset W_{ij}, \quad W_i W_{hj} = (0), \quad W_{hj} W_i = (0).$$

A Peirce torus yields a standard Peirce decomposition $N = \bigoplus_{i,j} e_i N e_j$ with respect to orthogonal $e_1, \ldots, e_m$ via the definitions $e_i N e_i = W_i, e_i N e_j = W_{ij}$ for distinct $i, j$.

(6.2) Theorem. The indecomposable nilpotent $k$-algebra $N$ accepts $S = k e_1 \oplus \cdots \oplus k e_s$ as nowhere trivial Wedderburn factor if and only if Der$_k N$ contains a Peirce torus of dimension $s - 1$. 


REFERENCES


DEPARTMENT OF MATHEMATICS, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CALIFORNIA 92115.