GLOBAL BIFURCATION THEOREMS
FOR NONCOMPACT OPERATORS

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1. Introduction. The first general existence theorem for bifurcation points was obtained by Krasnoselski [1]. He considered the equation
\[ u = \lambda L u + H(\lambda, u) \]
in a real Banach space \( B \) where \( L \) and \( H \) are compact, and \( H \) is \( o(||u||) \) uniformly on each bounded \( \lambda \) interval for small \( u \). In this situation he proved that if \( \lambda \) is a characteristic value of \( L \) having odd multiplicity, then \((\lambda, 0)\) is a bifurcation point in \( R \times B \). Much more recently, Rabinowitz [2] considered the same problem and, using a Leray-Schauder degree argument, obtained a two-fold alternative for the global behavior of these bifurcation branches.

This paper extends the results of Krasnoselski and Rabinowitz to a much larger class of operator equations. First to be considered is the equation
\[ Lu = \lambda u + H(\lambda, u) \]
in a real Hilbert space \( H \), where \( H \) is as above and \( L \) is selfadjoint (bounded or unbounded). In this case, each isolated eigenvalue of \( L \) having odd multiplicity is a bifurcation point possessing a continuous branch. Moreover, an alternative theorem on the global behavior of these branches is obtained.

By use of similar arguments these results for selfadjoint operators are extended to a general class of linear operators in a real Banach space \( B \).

2. The selfadjoint operators. In this section all work is in a real Hilbert space \( H \), \( L \) is a selfadjoint operator taking \( H \) into \( H \), and \( H(\lambda, u) \) is a compact operator taking \( R \times H \) into \( H \) that is \( o(||u||) \) uniformly on each bounded \( \lambda \) interval for small \( u \).

Let \( E \) denote \( R \times H \) with the product topology. For \( V \subset E \), a subcontinuum of \( V \) is a subset of \( V \) which is closed and connected in \( E \). The trivial solutions of (1) are the points \((\lambda, 0)\), and all other solutions are called nontrivial. Let \( S \) denote all nontrivial solutions of (1), and let \( E_{\lambda_0} \) denote the maximal subcontinuum of \( S \cup (\lambda_0, 0) \) containing \((\lambda_0, 0)\).

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For a subset $A$ of $R$, $H$, or $E$, $Cl(A)$ denotes its closure in the respective space. For $A \subseteq E$, $A_R$ denotes $\{\lambda | (\lambda, u) \in A \text{ for some } u\}$, and $A_H$ denotes $\{u | (\lambda, u) \in A \text{ for some } \lambda\}$. By an isolated eigenvalue $\lambda$ of $L$, we mean that $\lambda$ is an eigenvalue of $L$ and $\text{dist}(\lambda, \text{sp } L\setminus\lambda) > 0$.

The following lemma is stated without proof.

**Lemma 1.** Suppose $\lambda_0$ is an isolated eigenvalue of $L$ having finite multiplicity. Assume $\mathcal{C}_{\lambda_0}$ is bounded, $\text{Cl}(\mathcal{E}_{\lambda_0}) \cap \text{ess sp } L = \emptyset$, and $\mathcal{C}_{\lambda_0} \cap \{R \times \{0\}\} = (\lambda_0, 0)$. Then $\mathcal{C}_{\lambda_0}$ is compact and there exists a bounded open set $\mathcal{O} \subseteq E$ such that $\mathcal{C}_{\lambda_0} \subseteq \mathcal{O}$, $\partial \mathcal{O} \cap H = \emptyset$, $\text{Cl}(\mathcal{O}_H) \cap \text{ess sp } L = \emptyset$, the only trivial solutions contained in $\mathcal{O}$ are points $(\lambda, 0)$ where $|\lambda - \lambda_0| < \varepsilon$ for some $\varepsilon < \varepsilon_0 = \text{dist}(\lambda_0, \text{sp } L\setminus\lambda_0)$, and $\text{dist}(\partial \mathcal{O}, \{\text{sp } L \times \{0\}\}) \geq 2\varepsilon_1$ for some positive $\varepsilon_1$.

**Remark.** The theorem below will show that the hypotheses of the preceding lemma imply that $\lambda_0$ is an eigenvalue of even multiplicity.

**Theorem 1.** Let $\lambda_0$ be an isolated eigenvalue of $L$ having odd multiplicity. Then

(i) $\mathcal{C}_{\lambda_0}$ is unbounded, or

(ii) $\mathcal{C}_{\lambda_0}$ is bounded and $\text{Cl}(\mathcal{E}_{\lambda_0}) \cap \text{ess sp } L \neq \emptyset$, or

(iii) $\mathcal{C}_{\lambda_0}$ is compact, $\text{Cl}(\mathcal{E}_{\lambda_0}) \cap \text{ess sp } L = \emptyset$, and $\mathcal{C}_{\lambda_0}$ contains trivial solutions other than $(\lambda_0, 0)$.

**Proof.** Let us define $\Phi(\lambda, u) = Lu - \lambda u - H(\lambda, u)$. In general, degree theory cannot be applied to such an operator. Under the hypothesis on $L$ we will show how $\Phi$ can be replaced by a compact perturbation of the identity, thus allowing the use of degree theory.

Assume that none of (i), (ii), and (iii) occurs. Then by Lemma 1 we find a bounded open set $\mathcal{O}, \varepsilon > 0$, and $\varepsilon_1 > 0$, such that $\mathcal{C}_{\lambda_0} \subseteq \mathcal{O}$, $\text{Cl}(\mathcal{O}_H) \cap \text{ess sp } L = \emptyset$, $\partial \mathcal{O} \cap H = \emptyset$, $\text{dist}(\partial \mathcal{O}, \{\text{sp } L \times \{0\}\}) \geq 2\varepsilon_1$, and the only trivial solutions to (1) in $\mathcal{O}$ are points $(\lambda, 0)$ satisfying $|\lambda - \lambda_0| < \varepsilon < \varepsilon_0$, where $\varepsilon_0 = \text{dist}(\lambda_0, \text{sp } L\setminus\lambda_0)$.

Select a neighborhood $N$ of $\text{ess sp } L$ which contains $\text{Cl}(\mathcal{O}_H)$ in its exterior, and let $\mu_0 \notin \text{Cl}(\mathcal{O}_H)$ be in the resolvent set. Let $H' \supseteq H''$ denote the maximal closed subspace for which $H'' \subseteq H'$ and $\text{sp } L|H'' = \text{sp } L \cap N$, and let $P$ be the projector onto $H''$. Define the linear operator $L_0$ by

$$L_0 = (L - \mu_0 I)(I - P).$$

$L_0$ is clearly compact. Furthermore, $\lambda \notin N$ is an eigenvalue of $L$ having multiplicity $m$ if and only if $\lambda - \mu_0$ is an eigenvalue of $L_0$ having multiplicity $m$. For $\lambda \notin \{\mu_0\} \cup N$ we define

$$G_\lambda = (\lambda - \mu_0)^{-1}[L_0 + (I - P)(-H(\lambda, u))] + (\lambda - L)^{-1}P(-H(\lambda, u)).$$
From the definition of $P$ it follows that (1) is equivalent to

$$u = G\lambda u$$

for $\lambda$ in a neighborhood of $\text{Cl}((\mathcal{C}_R))$. The linear part of $G\lambda$ is compact and the linear part of $G_{\lambda_0}$ has the eigenvalue 1 with multiplicity $m_0$ if and only if $L$ has the eigenvalue $\lambda_0$ with multiplicity $m_0$. The nonlinear part of $G\lambda$ is also compact and in norm is $o(\|u\|)$ for small $u$.

(2) is the form necessary for the use of Leray-Schauder degree theory. Applying this theory as Rabinowitz [2] did shows that one of (i), (ii), or (iii) must occur.

**Remark.** If the multiplicity of $\lambda_0$ is odd, Theorem 1 guarantees that $A_0$ is a bifurcation point with a continuous branch $\mathcal{C}_{\lambda_0}$.

**Corollary 1.** Let $\lambda_0$ be an isolated eigenvalue of $L$ of finite multiplicity which is a bifurcation point with continuous branch $\lambda_0$. Then

(i)' $\mathcal{C}_{\lambda_0}$ is unbounded, or

(ii)' $\mathcal{C}_{\lambda_0}$ is bounded and $\text{Cl}((\mathcal{C}_{\lambda_0})_R) \cap \text{ess sp } L \neq \emptyset$, or

(iii)' $\mathcal{C}_{\lambda_0}$ is compact, $\text{Cl}((\mathcal{C}_{\lambda_0})_R) \cap \text{sp } L = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ is even.

We now consider

$$Lu = \lambda Ku + H(\lambda, u),$$

where $K$ is positive definite and bounded and $L$, $H$ are as above.

**Corollary 2.** Let $R$ be the positive square root of $K$. Let $\lambda_0$ be an isolated eigenvalue of $R^{-1}LR^{-1}$ of finite multiplicity which is a bifurcation point of (3) with a continuous branch $\mathcal{D}_{\lambda_0}$. Then

(i) $\mathcal{D}_{\lambda_0}$ is unbounded, or

(ii) $\mathcal{D}_{\lambda_0}$ is bounded and $\text{Cl}((\mathcal{D}_{\lambda_0})_R) \cap \text{ess sp }(R^{-1}LR^{-1}) \neq \emptyset$, or

(iii) $\mathcal{D}_{\lambda_0}$ is compact, $\text{Cl}((\mathcal{D}_{\lambda_0})_R) \cap \text{sp } (R^{-1}LR^{-1}) = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ (of $R^{-1}LR^{-1}$) is even.

If the multiplicity of $\lambda_0$ is odd, then $(\lambda_0, 0)$ is a bifurcation point possessing a continuous branch.

3. **General operators.** We now generalize by considering a real Banach space $\mathcal{B}$ and linear operators $T: \mathcal{B} \rightarrow \mathcal{B}$. The equation being studied is

$$Tu = \lambda u + H(\lambda, u)$$

with $H$ as before.
THEOREM 2. Suppose $\lambda_0$ is an isolated eigenvalue of $T$ of odd multiplicity and

(a) to every closed interval $\sigma \subset R \setminus \text{ess sp } T$ containing $\lambda_0$ there is a compact projector $Q_\sigma$ that commutes with $T$, and $\lambda_0$ is an isolated eigenvalue of $T|Q_\sigma B$ of odd multiplicity,

(b) the restriction of $T - \lambda I$ to $(I - Q_\sigma)B$ is invertible for $\lambda \in \sigma$.

Then $(\lambda_0, 0)$ is a bifurcation point possessing a continuous branch $C_{\lambda_0}$ such that

(i) $C_{\lambda_0}$ is unbounded, or

(ii) $C_{\lambda_0}$ is bounded and $\text{Cl}(\langle C_{\lambda_0} \rangle_R) \cap \text{ess sp } T \neq \emptyset$, or

(iii) $C_{\lambda_0}$ is compact, $\langle C_{\lambda_0} \rangle_R \cap \text{sp } T = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ is even.

PROOF. The proof is similar to that of Theorem 1.

COROLLARY 3. Suppose $\lambda_0$ is an isolated eigenvalue of $T$ of odd multiplicity and for every closed interval $\sigma \subset R \setminus \text{ess sp } T$ containing $\lambda_0$, $T$ can be uniformly approximated by operators $T_\varepsilon$ which are of the type treated in Theorem 2 and such that $\text{sp } T_\varepsilon \cap \sigma = \text{sp } T \cap \sigma$ up to multiplicity of eigenvalues. Then the results of Theorem 2 hold for $T_\varepsilon$ and $C_{\lambda_0}$.

Our work necessitates the use of a complexification of $B$ which is denoted by $\hat{B} = B \times B$. The general element of $\hat{B}$ is

$$(x, y) = x + iy \quad \text{and} \quad \|(x, y)\|_{\hat{B}} = (\|x\|^2 + \|y\|^2)^{1/2},$$

where $\| \cdot \|$ is the norm in $B$. For any linear $T: B \to B$, $\hat{T}: \hat{B} \to \hat{B}$ is its unique linear extension to $\hat{B}$.

THEOREM 3. Let $T$ be a bounded linear operator and $\sigma$ be a compact subset of $R \setminus \text{ess sp } \hat{T}$. Then there is a bounded projector $Q_\sigma$ that commutes with $T$ such that the restriction of $T - \lambda I$ to $(I - Q_\sigma)B$ is invertible for $\lambda \in \sigma$ and $Q_\sigma B$ is the span of the principal manifolds belonging to eigenvalues of $T$ in $\sigma$.

PROOF. The first step is to go to the complexifications $\hat{T}$ and $\hat{B}$. A decomposition theorem [3] is applicable to this complex case. From this complex decomposition, we can derive suitable real projections from $\hat{B}$ into $B$ and their corresponding subspaces in $B$.

REMARK. It follows from this theorem that Theorem 2 holds for all bounded linear operators $T$ on $B$ for which $R \cap \text{ess sp } \hat{T} = \text{ess sp } T$. In particular this is true if $T$ is compact, or if $B$ is a Hilbert space and $T$ is selfadjoint.
REFERENCES


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