$L^p$-CONVOLUTION OPERATORS AND TENSOR PRODUCTS OF BANACH SPACES

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Associated with every locally compact group are triples of Banach algebras

(1) \( \{\mathcal{C}_0(G), L^1(G), L^\infty(G)\}, \quad \{A(G), C^*(G), B(G)\} \)

intimately connected with duality theory (the notation is that of [1]). In both cases the middle algebra is the closure of \( L^1(G) \) in the dual of the first algebra and also the predual of the third algebra (at least when \( G \) is amenable in the second case). Furthermore, the third algebra is closely connected with the multiplier algebra of the first algebra.

For abelian groups, compact or discrete, Varopoulos [11], [12] showed to great effect how the second triple could be obtained and studied by starting with the tensor product \( \mathcal{C}_0(G) \otimes_{\gamma} \mathcal{C}_0(G) \), \( \gamma \) the greatest cross-norm. An analogous construction starting this time with \( \mathcal{C}_0(G) \otimes_{\lambda} \mathcal{C}_0(G) \), \( \lambda \) the least cross-norm, would produce the first triple. On the other hand, at least for amenable groups, the triples in (1) can be considered as the extreme case \( p=1, 2 \), respectively, of a family \( \{A^p(G), cv^p(G), B^p(G)\}, \)

\[ 1 \leq p \leq 2, \]

associated with \( L^p \)-convolution operator theory, and obtained by starting with the tensor product \( L^p(G) \otimes_{\gamma} L^p(G) \), \( p \neq 1 \), or \( \mathcal{C}_0(G) \otimes_{\gamma} L^1(G) \), \( p=1 \). Indeed, Herz has shown that \( A^p(G) \) is a pointwise Banach algebra [6] while \( B^p(G) \), \( 1 < p \leq 2 \), is both the multiplier algebra of \( A^p(G) \) and the Banach dual space of \( cv^p(G) \), \( G \) amenable. In these notes we outline a new approach to convolution operator theory, by starting with \( \mathcal{C}_0(G) \otimes_{\alpha} \mathcal{C}_0(G) \), \( \alpha \) a tensorial norm [5], rather than with \( L^p(G) \otimes_{\gamma} L^p(G) \). A triple \( \{N^p(G), L^{p\infty}(G), \mathcal{W}^{p\infty}(G)\} \) analogous to (1) is obtained. For \( L^p \)-convolution operator theory, a family of tensorial norms \( \alpha_{pq} \) is used. The two basic ideas are to exploit classical Banach space theory concerning \( L^p(\mu) \)-spaces, for example, forgetting about group structure, and, then, when a group structure is imposed, to exploit standard \( \mathcal{C}_0(G) \)- and \( L^1(G) \)-techniques because all the \('L^p\)-theory’ has been thrown into the norm \( \alpha_{pq} \),


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associated with which is a highly developed operator ideal theory (cf. [2], [4]). Solutions to a number of open problems are obtained (cf. §4). Detailed proofs will appear elsewhere.

1. Varopoulos spaces, algebras. By a Varopoulos space we shall mean any Banach space $V^a(X, Y)$ of the form $V^a(X, Y) = \mathcal{C}_0(X) \otimes \alpha \mathcal{C}_0(Y)$, where $X, Y$ are locally compact Hausdorff spaces and $\alpha$ a tensorial norm. The simplest such space is $V^a(X, Y) = \mathcal{C}_0(X \times Y)$, and there is always a norm-decreasing isomorphism $\sum_{\alpha}^a: V^a(X, Y) \to V^a(X, Y)$. Thus

$$\mathcal{C}_0(X) \otimes \mathcal{C}_0(Y) = V^a(X, Y) \subseteq V^a(X, Y) \subseteq V^a(X, Y) = \mathcal{C}_0(X \times Y).$$

A Varopoulos space $V^a(X, Y)$ is said to be a Varopoulos algebra if $\alpha(f \cdot g) \leq \alpha(f) \alpha(g)$ for the pointwise product of $f, g \in \mathcal{C}_0(X) \otimes \mathcal{C}_0(Y)$ ($\alpha(\cdot)$ norm on $V^a(X, Y)$).

**Theorem 1.** Each Varopoulos algebra $V^a(X, Y)$ is a commutative semisimple Banach algebra with maximal ideal space $X \times Y$. Furthermore, $V^a(X, Y)$ is regular and selfadjoint.

Both $V^a(X, Y)$ and $V^a(X, Y)$ are Varopoulos algebras, and any $V^a(X, Y)$ is a Banach $V^a(X, Y)$-module. More generally, for any sequence $\{x_n\}$ (finite or infinite) in any Banach space $X$, set

$$M_r(\{x_n\}) = \sup \left\{ \left( \sum_n |\langle x^*, x_n \rangle|^r \right)^{1/r} : x^* \in X^*, \|x^*\| \leq 1 \right\}, \quad r \neq \infty,$$

$$(M_\infty(\{x_n\}) = \sup_n \|x_n\|).$$ When $1 \leq q \leq p \leq \infty$ and $t \in X \otimes Y$ set

$$\alpha_{pq}(t) = \alpha_{pq}(t; X, \mathcal{Y}) = \inf \left( \sum_n |\lambda_n|^r \right)^{1/r} \left( M_p(\{x_n\}) M_q(\{y_n\}) \right),$$

where $1/q + 1/q' = 1, 1/p + 1/q' + 1/r = 1$ and the infimum is taken over all representations $t = \sum_n \lambda_n x_n \otimes y_n$ (cf. [8]). If $X = \mathcal{C}_0(X)$, then

$$M_r(\{f_n\}) = \sup \left\{ \left( \sum_n |f_n(x)|^r \right)^{1/r} : x \in X \right\},$$

and so we deduce

**Theorem 2.** The Varopoulos space $V^{pq}(X), 1 \leq p \leq q \leq \infty$, obtained by taking $\alpha = \alpha_{pq}$, is always a Varopoulos algebra.

2. Fundamental properties. Deep Banach space results yield properties of $V^{pq}(X, Y)$. For any pair $X, Y$

$$\mathcal{C}_0(X) \otimes \mathcal{C}_0(Y) = V^{1\infty}(X, Y) \subseteq V^{pq}(X, Y) \subseteq V^{11}(X, Y) = \mathcal{C}_0(X \times Y)$$

isometrically or with norm-decreasing inclusion. The right-hand equality
follows from the fact that $\mathcal{C}_0(Y)$ is an $L_{1+\varepsilon}$-space for every $\varepsilon > 0$ [9]. From the Grothendieck ‘fundamental theorem for metric spaces’ [5], [9] we obtain

**Theorem 3.** Up to equivalence of norms, $V^{2\gamma}(X, Y) = \mathcal{C}_0(X) \otimes \gamma \mathcal{C}_0(Y)$; in fact,

$$\alpha_{2\gamma}(f) \leq \gamma(f) \leq K_G \alpha_{2\gamma}(f), \quad f \in \mathcal{C}_0(X) \otimes \mathcal{C}_0(Y),$$

where $K_G$ is the Grothendieck universal constant.

**Theorem 4 (Kwapień-Pietsch).** A linear operator $T: \mathcal{C}_0(Y) \to M(X)$ belongs to $(V^{2\gamma}(X, Y))^*$ if and only if for each $\varepsilon > 0$ there exist probability measures $\mu$ on $X$ and $\nu$ on $Y$ such that

$$|\langle f, Tg \rangle| \leq (1 + \varepsilon) \|T\|_{pq} \left( \int_X |f|^p d\mu \right)^{1/p} \left( \int_Y |g|^q d\nu \right)^{1/q}$$

for all $f \in \mathcal{C}_0(X)$, $g \in \mathcal{C}_0(Y)$.

Use of the bilinear Riesz-Thorin theorem and Theorem 3 now gives

**Theorem 5.** (i) If $1 \leq p \leq 2$ and $2 \leq q \leq \infty$, then $V^{2\gamma}(X, Y) = V^\gamma(X, Y)$ up to equivalence of norms.

(ii) If $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$, then $V^{rs}(X, Y) \subseteq V^{pq}(X, Y)$ where

$$\frac{1}{r} = \frac{1 - \theta}{2} + \frac{\theta}{p}, \quad \frac{1}{s} = \frac{1 - \theta}{2} + \frac{\theta}{q}, \quad 0 < \theta < 1,$$

the embedding being continuous.

(iii) If $1 \leq p < q < 2$, then $V^{pq}(X, Y) = V^{1q}(X, Y)$ up to equivalence of norms.

The proof of property (iii) in Theorem 5 also uses the fact that a bounded linear operator $T: \mathcal{C}_0(Y) \to L^1(\mu)$ automatically is absolutely $s$-summing if $2 < r < s \leq \infty$ (cf. [10]).

3. The triple $\{\mathcal{V}^s(G), \mathcal{L}^s(G), \mathcal{H}^s(G)\}$. Let $G$ be a locally compact group, and $L^1(G) \otimes \delta L^1(G)$ the completion of $L^1(G) \otimes L^1(G)$ with respect to the associate norm $\cdot^\delta$ of $\cdot$ [5]; equivalently, $L^1(G) \otimes \delta L^1(G)$ is the closure of $L^1(G \times G)$ in $(V^s(G, G))^*$. Starting from $\mathcal{C}_0(G) \otimes \mathcal{C}_0(G)$, $L^1(G) \otimes \delta L^1(G)$, $(L^1(G) \otimes \delta L^1(G))^*$, we define a triple $\{\mathcal{V}^s(G), \mathcal{L}^s(G), \mathcal{H}^s(G)\}$. Now from Theorem 3 it follows (nontrivially!) that $(L^1(G) \otimes \delta L^1(G))^*$ always contains $M(A(G))$ where $(M\bar{\phi})(x, y) = \bar{\phi}(xy^{-1})$.

**Definition 1.** $\mathcal{V}^s(G)$ will denote the completion of $A(G)$ with respect to the norm induced on $M(A(G))$ by $(L^1(G) \otimes \delta L^1(G))^*$.

Clearly

$$A(G) \subseteq \mathcal{V}^{\gamma}(G) \subseteq \mathcal{V}^{s}(G) \subseteq \mathcal{V}^{\lambda}(G) = \mathcal{C}_0(G).$$
The closure of $L^1(G)$ in $(\mathcal{W}^A(G))^*$ is denoted by $L^{p^*}(G)$. Then

$$L^1(G) = L^{p^*}(G) \subseteq L^{p^*}(G) \subseteq C^*(G).$$

Finally, set $\mathcal{W}^A(G) = \{ \phi \in L^{\infty}(G): M(\phi) \in (L^1(G) \otimes_{\alpha} L^1(G))^* \}$; clearly

$$B(G) \subseteq \mathcal{W}^A(G) \subseteq \mathcal{W}^A(G) \subseteq \mathcal{W}^A(G) = L^\infty(G).$$

When $\alpha = \alpha_{p',q'}$, $1 \leq p \leq q \leq \infty$, we write $\mathcal{W}^{pq}(G)$, $\mathcal{W}^{pq}(G)$.

**Theorem 6.** If $V^A(G,G)$ is a Varopoulos algebra, then $\mathcal{W}^A(G)$ and $\mathcal{W}^A(G)$ are Banach algebras under pointwise multiplication. In particular, $\mathcal{W}^A(G)$ always is such a Banach algebra. For arbitrary tensorial norm $\alpha$, $\mathcal{W}^A(G)$ and $\mathcal{W}^A(G)$ are Banach $\mathcal{W}^A(G)$-modules while

$$\mathcal{W}^A(G) \cap C(G) = \mathcal{W}^A(G) \cap C(G) = \text{Bochner-Eberlein}$$

isometrically ($G_d = G$ with discrete topology).

The most precise results are obtained when $G$ is amenable with an interesting use of the Glicksberg-Reiter theorem.

**Theorem 7.** Let $G$ be an amenable group. Then, up to equivalence of norms,

$$\mathcal{W}^A(G) = (L^{p^*}(G))^*, \quad \mathcal{W}^A(G) \cap C(G) = (L^{p^*}(G))^* \cap C(G),$$

where $L^{p^*}(G)$ denotes the closure of $l^1(G_d)$ in $(\mathcal{W}^A(G))^*$.

4. **Applications to $L^p$-convolution operator theory.** Using characterizations of $(C_0(X) \otimes C_0(Y))^*$ in terms of $(p,q)$-absolutely summing operators stemming from Theorem 4, together with characterizations of $(L^1(G) \otimes_{\alpha} L^1(G))^*$, $\alpha = \alpha_{p',q'}$, in terms of $(p,q)$-integral operators, we obtain

**Theorem 8.** For any locally compact group $G$ the following inclusions hold:

(i) $A^p(G) \subseteq \mathcal{W}^{pq}(G)$, $1 \leq p \leq \infty$;

(ii) $B(G) \subseteq \mathcal{W}^{pq}(G) \subseteq B^p(G)$, $1 < p < \infty$;

except possibly for the first inclusion in (ii) all embeddings are norm-decreasing.

Theorems 2, 5 and 6 now provide a completely new approach to the main results of Herz (both in [6] and unpublished) since $\mathcal{W}^{pq}(G)$ is a closed subspace of $\mathcal{W}^{pq}(G)$.

**Theorem 9.** For each locally compact group $G$ and each $p$, $1 \leq p \leq \infty$, $A^p(G)$ is a Banach algebra and a Banach $A^p(G)$-module via pointwise multiplication when $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq \infty$. 

These algebras $\mathcal{V}^{pq}(G)$ have useful identifications. The Banach space of (right-) convolution operators $T: L^r(G) \to L^s(G)$ will be denoted by $C_{sv}(G)$, the closure of $L^1(G)$ in $C_{sv}(G)$ by $C_{sv}(G)$ (when $r, s$ are suitably restricted). In case $r = \infty$ or $s = \infty$, $C_{sv}(G)$ replaces $L^\infty(G)$. It is known that, when $G$ is amenable, $C_{sv}(G) = (A^{pq}(G))^*$ isometrically when

$$A^{pq}(G) = P(L^r(G) \otimes L^s(G))$$

(cf. [1]).

**Theorem 10.** Let $G$ be an amenable group. Then isometrically

(i) $A^p(G) = \mathcal{V}^{pq}(G)$, $1 \leq p \leq \infty$;
(ii) $C^p(G) = \mathcal{L}^{pq}(G)$, $1 \leq p \leq \infty$;
(iii) $B^p(G) = \mathcal{V}^{pq}(G)$, $1 < p < \infty$;
(iv) $(C^p(G))^* = B^p(G)$, $1 < p < \infty$.

In particular,

$$A(G) \subseteq A^p(G) \subseteq A^p(G) \subseteq C_0(G)$$

whenever $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq \infty$.

**Theorem 11.** Let $G$ be a compact group. Then isometrically

(i) $A^{pq}(G) = \mathcal{V}^{pq}(G)$,
(ii) $C^{pq}(G) = \mathcal{L}^{pq}(G)$

for $1 \leq p \leq q \leq \infty$.

Part (iii) of Theorem 5 when translated into convolution operator theory says that

$$C^{pq}(G) = C^{pq}(G), \quad G \text{ compact}, \quad 1 \leq r < p < 2.$$

Doss established (2) for compact abelian groups by showing that $C^{pq}(G)$ coincides with the space $C^{pq}(G)$ of weak type $(p, p)$ convolution operators. Setting $A^{pq}(G) = P(L^r(G) \otimes L^s(G))$, $G$ abelian, $1 < p < \infty$, we can complete the identification of the algebras $\mathcal{V}^{pq}(G)$, $G$ abelian.

**Theorem 12.** Let $G$ be a locally compact abelian group. Then

$$\mathcal{V}^{pq}(G) = A^p(G), \quad \mathcal{L}^{pq}(G) = C^p(G), \quad (\mathcal{V}^{pq}(G))^* = C^p(G)$$

provided $1 \leq r < p < 2$. In particular, $A^p(G)$ is a pointwise Banach algebra.

The techniques which this approach provides lead to solutions for arbitrary amenable groups of many of the problems left open by Eymard [1]. In addition, nine of the squares left open in the multiplier table given by Hewitt-Ross [7, pp. 410–411] can be completed and partial information given for the remaining two.
REFERENCES


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