OBSTRUCTIONS TO TRANSVERSALITY
FOR COMPACT LIE GROUPS

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Throughout G is a compact Lie group which is topologically cyclic
with dense generator g. Let N and M be smooth G manifolds without
boundary and Y ⊂ M a closed invariant submanifold. All manifolds are
oriented and G preserves orientation. Let f: N → M be a proper G map.
When is f properly G homotopic to a map γ which is transverse regular to
Y ⊂ M, written γ ∩ Y? We introduce obstructions which show that trans-
versality is a global phenomena in contrast to the case G = 1 where every-
thing is local and trivial.

Without loss of generality, we may assume that f^\# : N^\# → M^\# is transverse
to Y^\# and set X^\# = (f^\#)^{-1}(Y^\#). For each oriented real G vector bundle v
over X^\# such that the G representation on each fiber of v has no trivial
factor and g preserves orientation on each fiber, let λ_\pm(v) be the ± eigen-
bundles of the canonical involution \tau on \hat{\lambda}(v \otimes C) = \sum \hat{\lambda}(v \otimes C)
constructed from the orientation and an inner product on v. Let λ_\pm(v \otimes C) =
\sum (-1)^i \hat{\lambda}(v \otimes C), I^{X^\#} ∈ K_G(TX^\#) be the index class of X^\#, i.e. the symbol
of the operator D^+. See [1, p. 575]. Let \mathcal{P} ⊂ R(G) be the prime ideal of
characters \{X ∈ R(G)| X(g) = 0\} and

(i) \mathcal{B}(v) = \frac{\lambda_+(v) - \lambda_-(v)}{\lambda_-(v \otimes C)} I^{X^\#} ∈ K_G(TX^\#)_{\mathcal{P}}.

Let f: X → Y be a G map. If f is an embedding there is a homomorphism
f^!: K_G(TX) → K_G(TY) [1]. By taking the product of Y with a real G module
and using the Thom isomorphism for complex G vector bundles, we may
assume that f^! is defined for any map f and denote it by f^!. The normal
bundle of Y in M is denoted by ν(Y, M). Its restriction to Y^\# has a splitting

(ii) (i^\#)^* ν(Y, M) = ν(Y, M)^\# + ν_\#(Y, M),

where ν(Y, M)^\# is the subbundle of points fixed by g and i^\#: Y^\# → Y is


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the inclusion. Let $j^\varphi: X^\varphi \to N^\varphi$ be the inclusion and define $\nu = \nu(f)$ by

\begin{equation}
(iii) \quad \nu + (f^\varphi)^*\nu_\varphi(Y, M) = (j^\varphi)^*\nu(N^\varphi, N)
\end{equation}

and set

$$B_G = \mathscr{B}(\nu(f)) \in K_G(TX^\varphi).$$

The inclusions of $TN^\varphi$ in $TN$ and $TY^\varphi$ in $TY$ are denoted by $Th^\varphi$ and $Ti^\varphi$. Let $(1dG_{X^\varphi})_{\varphi}: K_G(TX^\varphi_{\varphi}) \to R(G_{\varphi})$ be the localization of the Atiyah-Singer index homomorphism. The group of connected components of $G$ is denoted by $\Pi_0(G)$. Define

\begin{equation}
(iv) \quad g(f) = (1dG_{X^\varphi})_{\varphi}(B_G) \in R(G_{\varphi})/R(\Pi_0(G)),
\end{equation}

\begin{equation}
(v) \quad l(f) = \lambda_{-1}(\nu(N^\varphi, N) \otimes C) \cdot j^\varphi_{*}(B_G) \in K_G(TN^\varphi)_{\varphi}/(Th^\varphi)^*K_G(TN),
\end{equation}

\begin{equation}
(vi) \quad \vartheta(f) = \lambda_{-1}(\nu(Y^\varphi, Y) \otimes C) f^\varphi_{*}(B_G) \in K_G(TY^\varphi)_{\varphi}/(Ti^\varphi)^*K_G(TY).
\end{equation}

**Theorem 1.** If $f: N \to M$ is properly $G$ homotopic to $\gamma$ and $\gamma \cap Y$, then $g(f)$, $l(f)$ and $\vartheta(f)$ are zero.

**Proof.** Suppose $f \cap Y$ and $X = f^{-1}(Y)$. Then $g(f) = \text{Sign}(G, X) \in R(\Pi_0(G))$; moreover, $\vartheta(f) = (Ti^\varphi)^*f^\varphi_{*}(I^X_{\varphi})$, where $I^X_{\varphi} \in K_G(TX)$ is the index class of $X$. Similarly one sees that $l(f) = 0$.

The notion of fiber homotopy equivalence is extended to the category of $G$ vector bundles as follows: Let $N$ and $M$ be two (real) $G$ bundles over a $G$ space $Y$. A $G$ map $\omega: N \to M$ is called a quasi-equivalence if $\omega$ is proper, fiber preserving and degree 1 on fibers. The notion of normal map is extended to the category of smooth, closed $G$ manifolds as follows: A *normal* $G$ map $f: X \to Y$ consists of a triple $[X, f, F]$ where $f: X \to Y$ is a $G$ map of degree 1 and $F$ is a bundle map $F: TX + f^*(N) \to TY + M$ covering $f$ for some pair of $G$ bundles $N$ and $M$ over $Y$. When $G = 1$ the set of normal cobordism classes of normal maps to $Y$ and the set of fiber homotopically equivalent bundles with appropriate equivalence relation are in 1-1 correspondence. Transversality provides the correspondence. For general $G$ and quasi-equivalence $\omega: N \to M$, the obstructions $g(\omega)$, $l(\omega)$ and $\vartheta(\omega)$ to making $\omega \cap Y$ give obstructions to converting a quasi-equivalence to a normal $G$ map.

**Example 1.** Let $G = S^1$ with dense generator $t$. Let $N$ and $M$ be two complex $S^1$ vector bundles over a closed $S^1$ manifold $Y$. To simplify the formula, we assume $N^t = M^t = Y$. The restrictions $\tilde{N}$ and $\tilde{M}$ of $N$ and $M$ to $Y^t$ have real splittings $\tilde{N} = \sum_{n > 0} N_n$, $\tilde{M} = \sum_{n > 0} M_n$ where, e.g., $N_n$ is the subbundle on which $t$ acts by multiplication by $t^n$. Similarly, $\nu(Y^t, Y)$ has such a splitting. Let

\begin{equation}
(vii) \quad A(t, \tilde{N}) = \prod_{n > 0} \prod_j \frac{t^n e^{aj} + 1}{t^n e^{aj} - 1} (N_n) \in H^*(Y^t, C),
\end{equation}
where the elementary symmetric functions of the $x_j = x_j(N_n)$ are the Chern classes of $N_n$,

$$(viii) \quad L'(TY^t) = \prod \frac{x_i}{\tanh(x_i/2)} (TY^t) \in H^*(Y^t, C),$$

where the elementary symmetric functions of the $x_i^2$ are the Pontrjagin classes of $Y^t$.

The ring $R(S^1)_\varphi$ is contained in the field $Q(t)$ of rational functions of $t$. The obstruction $g(\omega) \in Q(t) | Z (Z = R(1))$ is given by the rational function

$$(ix) \quad g(\omega)(t) = \left\langle A(t, \nu(Y^t, Y)) \frac{A(t, \tilde{\mathcal{N}})}{A(t, \tilde{\mathcal{M}})} L'(TY^t), [Y^t] \right\rangle$$

where $\langle \alpha, [Y^t] \rangle$ denotes evaluation of the cohomology class $\alpha$ on the orientation class $[Y^t] \in H_*(Y^t, C)$. Observe that the obstruction $g(\omega)$ does not depend on $\omega$. Essentially the reason is that $S^1$ is connected. (Compare (ix) with [1, (7.7)].)

**Example 1.** As a very special illustration of (ix), take $Y$ to be a point and $N$ and $M$ the complex two-dimensional $S^1$ modules $N = t^p + t^q$ and $M = t + t^{pq}$, $(p, q) = 1$, where $t \in S^1$ acts with eigenvalues $t^p$ and $t^q$, respectively, $t$ and $t^{pq}$. We view $N$ and $M$ as $S^1$ vector bundles over $Y$. Choose positive integers $a$ and $b$ such that $-ap + bq = 1$. Let $z = (z_0, z_1)$ be complex coordinates of a point $z \in N$ and set $\omega_0(z) = (z_0^a z_1^b, z_0^b + z_1^p)$. Then $\omega_0$ is a proper $S^1$ map and has degree 1; moreover,

$$(x) \quad g(\omega_0)(t) = \frac{(t^p + 1)(t^q + 1)(t^p - 1)(t^q - 1)}{(t^p - 1)(t^q - 1)(t + 1)(t^{pq} + 1)} \in \frac{Q(t)}{Z}.$$

**Product Lemma 3.** Let $N$ and $M$ be two complex $G$ modules viewed as $G$ bundles over a point, and $\omega: N \to M$ a quasi-equivalence. Then $\omega$ induces a quasi-equivalence $\tilde{\omega}: Y \times N \to Y \times M$ for any closed $G$ manifold $Y$ and

$$g(\tilde{\omega}) = \text{Sign}(G, Y) \cdot g(\omega).$$

**Corollary 4.** Let $Y$ be a closed $S^1$ manifold with $\text{Sign}(Y) = \text{Sign}(1, Y) \neq 0$. Let $\omega_0: N \to M$ and $\omega_0: Y \times N \to Y \times M$ be as above. Then $g(\tilde{\omega}_0) \neq 0$.

**Example 2.** Let $\omega: N \to M$ be a quasi-equivalence of $G$ bundles over $Y$. Assume $N = M = Y^q$ consists of $q$ isolated points. Then $K_G(TY^q)_\varphi = \prod_{j=1}^q R(G)_\varphi$ and the jth component of $\varphi(\omega)$ is

$$\varphi(\omega)_j = \frac{\chi_j(M_j \otimes C) \cdot \lambda_j(TY^q) - \lambda_j(TY^q)}{\lambda_1(N_j \otimes C) \cdot \lambda_j(M_j) - \lambda_j(M_j)}.$$
where $M_j$, $N_j$ and $TY_j$ denote the representations of $G$ defined by restricting $N$, $M$ and $TY$ to the $j$th isolated fixed point.

In order to illustrate ideas for closed manifolds, observe that any quasi-equivalence $\omega : N \to M$ induces a $G$ map $\omega^+ : N^+ \to M^+$ of the one point compactifications. In particular, take the $N$ and $M$ of Example 1' and $\omega = \omega_0$. Then $N^+$ and $M^+$ are smooth 4 spheres. Take $Y = (M^+)^{S^1}$. One finds that $\mathcal{O}(\omega_0^\dagger) \neq 0$.

**Contributions of subgroups $H \subset G$.** Each subgroup of $G$ can be used to generate new obstructions via the following observation: If $f : N \to M$ is transverse to $Y \subset M$ then $f^H : N^H \to M^H$ is transverse to $Y^H$ for each $H \subset G$; moreover, $f^H$ is a $G/H$ map. This means that if $\alpha_G$ is any transversality obstruction defined for all topologically cyclic groups $G$, then $\alpha_{G/H}(f^H)$ is a transversality obstruction for the $G$ map $f$, i.e., $\alpha_{G/H}(f^H) = \mathcal{O}_G(f)$ is an obstruction for $f$. Actually each component of $M^H$ contributes an obstruction.

**Example 3.** $G = S^1$. Let $\Omega = t^p + t^p + t^p + t^0$ be the complex 4-dimensional $S^1$ module where $t \in S^1$ acts with eigenvalues $t^p$, $t^p$, $t^p$ and $t^0$. Let $Y = P(\Omega)$ be the space of complex lines in $\Omega$. Then $Y$ is an $S^1$ manifold in an obvious way and if $\omega_0 : N \to M$ is the quasi-equivalence of Example 1', then $\omega_0 : Y \times N \to Y \times M$ and $g_{S^1}(\omega_0) = 0$ by the Product Lemma. On the other hand, $(Y \times M)^Z$ consists of two components and each contributes an obstruction

$$g_{S^1/Z_p}(\tilde{\omega}_0 Z)^i = \frac{t^1 + 1 \cdot t^0 - 1}{t^1 - 1 \cdot t^0 + 1} \in \mathbb{Q}(t)$$

for $i = 1, 2$. Here $S^1/Z_p$ is identified with $S^1$ with representation ring $Z[t, t^{-1}]$.

Equation (iii) provides the basis for an obstruction theory for $G$ transversality, but this and other details of $G$ transversality including the case of finite isotropy groups will appear elsewhere.

**Bibliography**