THE SPACE OF CLASS \( \alpha \) BAIRE FUNCTIONS

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Communicated by Jacob Feldman, December 6, 1973

ABSTRACT. Let \( X, Y \) be compact Hausdorff spaces and \( \mathcal{B}_\alpha^*(X) \), \( \mathcal{B}_\beta^*(Y) \), \( 0 \leq \alpha, \beta \leq \Omega \) (the first uncountable ordinal), the associated Banach spaces of bounded real-valued Baire functions of classes \( \alpha \) and \( \beta \). If \( \mathcal{B}_\alpha^*(X) \neq \mathcal{B}_\beta^*(Y) \) (which is the case if \( \alpha \neq \beta \) and \( X \) is not dispersed), then \( \mathcal{B}_\alpha^*(X) \) is neither linearly isometric to \( \mathcal{B}_\beta^*(Y) \) nor equivalent to \( \mathcal{B}_\beta^*(Y) \) in several other ways. \( \mathcal{B}_\alpha^*(X) \) is linearly isometric to \( \mathcal{B}_\beta^*(Y) \) if and only if \( X \) is Baire isomorphic to \( Y \). For \( 1 \leq \alpha < \Omega \) the maximal ideal space of \( \mathcal{B}_\alpha^*(X) \) for a nondispersed compact space \( X \) is not an \( F \)-space.

1. Let \( X \) be a compact (more generally, completely regular) Hausdorff space and \( C(X) \) the space of continuous real-valued functions on \( X \). Let \( \mathcal{B}_0(X) = C(X) \), and inductively define \( \mathcal{B}_\alpha(X) \) for each ordinal \( \alpha \leq \Omega \) (\( \Omega \) denotes the first uncountable ordinal) to be the space of pointwise limits of sequences of functions in \( \bigcup_{\xi < \alpha} \mathcal{B}_\xi(X) \). Let \( \mathcal{B}_\alpha^*(X) \) be the space of bounded functions contained in \( \mathcal{B}_\alpha(X) \). With the pointwise operations \( \mathcal{B}_\alpha(X) \) and \( \mathcal{B}_\alpha^*(X) \) are lattice-ordered algebras. With the supremum norm \( \mathcal{B}_\alpha^*(X) \) is a Banach algebra (see [4, §41]).

The Baire sets of \( X \) of multiplicative class \( \alpha \), denoted by \( Z_\alpha(X) \), are defined to be the zero sets of functions in \( \mathcal{B}_\alpha^*(X) \). Those of additive class \( \alpha \), denoted by \( CZ_\alpha(X) \), are defined as the complements of sets in \( Z_\alpha(X) \). Finally, those of ambiguous class \( \alpha \), denoted by \( A_\alpha(X) \), are the sets which are simultaneously in \( Z_\alpha(X) \) and \( CZ_\alpha(X) \). With the set-theoretic operations of union and intersection, \( A_\alpha(X) \) is a Boolean algebra for each \( \alpha \leq \Omega \). The sets of exactly ambiguous class \( \alpha \), denoted by \( EA_\alpha(X) \), are those in \( A_\alpha(X) \cap \bigcup_{\xi < \alpha} A_\xi(X) \). The sets of exactly additive and exactly multiplicative class \( \alpha \) are defined analogously. The class of all Baire subsets of \( X \) is \( Z_\Omega(X) \).


Key words and phrases. Banach space, Baire function, Baire class.

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A topological space is called realcompact if it is homeomorphic to a closed subset of a product of real lines.

**Theorem 1.** If $X$ and $Y$ are compact (more generally, realcompact) spaces, then every Boolean algebra isomorphism $f$ of $A_\alpha(X)$ onto $A_\beta(Y)$, $1 \leq \alpha, \beta \leq \Omega$, is induced by a point map $\phi$ of $X$ onto $Y$; that is, there exists a one-to-one map $\phi$ of $X$ onto $Y$ such that $\phi[B] = f(B)$ for each $B \in A_\alpha(X)$.

**Proof Outline.** Consider the compact set, denoted by $X_\alpha$, of non-zero multiplicative linear functionals on $B^*_\alpha(X)$ with the weak star topology. It follows from the fact that for each pair of disjoint sets $B_1, B_2 \in Z_\alpha(X)$, there is an $A \in A_\alpha(X)$ with $B_1 \subseteq A \subseteq X\setminus B_2$, that $X_\alpha$ has a base of clopen (closed and open) sets. Since the Boolean algebra of clopen sets of $X$ is isomorphic to $A_\alpha(X)$, the Stone space of $A_\alpha(X)$ is homeomorphic to $X_\alpha$.

The canonical embedding of $X$ into $X_\alpha$ which assigns a point in $X$ to the evaluation functional at that point maps $X$ onto a dense subset of $X_\alpha$. The induced topology on $X$ from $X_\alpha$ is discrete if and only if every point in $X$ is a $G_\delta$. The space $X_\alpha$ may thus be considered as a compactification of $X$ with the topology having $Z_0(X)$ as a base. From this point of view, each $f \in B^*_\alpha(X)$ has a unique extension to a $\hat{f} \in C(X_\alpha)$, and the map $\Phi: B^*_\alpha(X) \to C(X_\alpha)$ defined by $\Phi(f) = \hat{f}$ is an algebra isomorphism onto $C(X_\alpha)$. Details concerning the space $X_\alpha$ are contained in [5].

A filter $F$ of sets in $A_\alpha(X)$ ($Z_\alpha(X)$) is said to have the CIP (countable intersection property) if for each countable family $\{C_n\} \subseteq F$ there is a $C \in A_\alpha(X)$ (respectively $Z_\alpha(X)$) such that $C \subseteq \bigcap_{n=1}^{\infty} C_n$. A filter $F$ in $A_\alpha(X)$ ($Z_\alpha(X)$) is said to be fixed if $\bigcap\{F: F \in F\} \neq \emptyset$.

For any completely regular spaces $X$ and $Y$, if $f$ is a Boolean algebra isomorphism of $A_\alpha(X)$ onto $A_\beta(Y)$ and $M$ is a maximal filter in $A_\alpha(X)$ with the CIP, then $f[M]$ is a maximal filter in $A_\beta(Y)$ with the CIP.

If $X$ is realcompact, then every maximal filter with the CIP in $A_\alpha(X)$, $\alpha \geq 1$, is fixed. To see this let $M \subseteq A_\alpha(X)$ be a maximal filter with the CIP. Then, since each element of $Z_\alpha(X)$ is the countable intersection of elements in $A_\alpha(X)$, $M_{\delta}$ (the family of countable intersections of sets in $M$) is a maximal filter with the CIP in $Z_\alpha(X)$. Thus, since $X$ is realcompact and each set in $Z_\alpha(X)$ is obtainable from $Z_\alpha(X)$ by Souslin's operation $(A)$, $M_\delta$ is fixed. Thus $M$ is fixed. Here we have used the following set-theoretic result due to Z. Frolik [2]: Let $H_1$ and $H_2$ be families of subsets of a set
$X$ which are closed under countable intersections, and let $\mathcal{M}(H_i)$, $i = 1, 2$, denote the set of free maximal filters in $H_i$ with the CIP. If $H_1 \subseteq H_2$ and every $H \in H_2$ is a Souslin-$H_1$ set (that is, can be represented in the form

$$H = \bigcup_{i_1, i_2, \cdots} \bigcap_{n=1}^{\infty} H_{i_1, \cdots, i_n}, \quad H_{i_1, \cdots, i_n} \in H_1$$

where the union is over all sequences of positive integers $(i_1, i_2, \cdots)$, then the map $\mathcal{M} \to \mathcal{M} \cap H_1$, $\mathcal{M} \in \mathcal{M}(H_2)$ is one-to-one onto $\mathcal{M}(H_1)$.

From this it follows that if $X$ and $Y$ are realcompact, $\alpha, \beta \geq 1$, and $f$ is a Boolean algebra isomorphism of $A_\alpha(X)$ onto $A_\beta(Y)$, then the induced homeomorphism $\phi$ of their Stone spaces, namely $X_\alpha$ and $Y_\beta$, maps $X$ onto $Y$; that is, $\phi[X] = Y$. Also for each $B \in A_\alpha(X)$, $\phi[B] = f(B)$. This completes the proof.

2. Let $X$ and $Y$ be completely regular spaces. A Baire isomorphism of class $(\alpha, \beta; \gamma, \delta)$ of $X$ onto $Y$ is a one-to-one map $f$ of $X$ onto $Y$ such that

$$f[Z_\alpha(X)] \subseteq Z_\beta(Y) \quad \text{and} \quad f^{-1}[Z_\delta(Y)] \subseteq Z_\gamma(X).$$

**Theorem 2.** If $X$ and $Y$ are compact (more generally, realcompact) spaces and $\alpha, \beta \geq 1$, then the following are equivalent:

1. There exists a Baire isomorphism of class $(\alpha, \beta; \alpha, \beta)$ of $X$ onto $Y$.
2. $B_\alpha^*(X)$ is linearly isometric to $B_\beta^*(Y)$.
3, 4, 5, 6 $B_\alpha^*(X)$ is isometric (ring, lattice, multiplicative semigroup isomorphic) to $B_\beta^*(Y)$.
6, 7, 8, 9 $B_\alpha(X)$ is ring (lattice, multiplicative semigroup) isomorphic to $B_\beta(Y)$.

**Proof outline.** (2) $\Rightarrow$ (1). Since $B_\alpha^*(X)$ and $B_\beta^*(Y)$ are linearly isometric to $C(X_\alpha)$ and $C(Y_\beta)$ respectively, $X_\alpha$ and $Y_\beta$ are homeomorphic. By Theorem 1 such a homeomorphism induces a Baire isomorphism of class $(\alpha, \beta; \alpha, \beta)$ of $X$ onto $Y$.

All of the other nontrivial implications follow similarly.

**Remark.** Since for a completely regular space $X$ every $f \in B_\alpha^*(X)$ ($B_\alpha(X)$) has a unique extension to a $\hat{f} \in B_\alpha^*(\nu X)(B_\alpha(X))$, where $\nu X$ denotes the Hewitt realcompactification of $X$ [6], it follows from Theorem 2 that for completely regular spaces $X$ and $Y$, parts (2) through (9) of Theorem 2 are
equivalent, and these are equivalent to the existence of a Baire isomorphism of class \((\alpha, \beta; \alpha, \beta)\) of \(\nu X\) onto \(\nu Y\). More generally yet, Theorems 1 and 2 may be phrased in terms of zero-set spaces and suitably defined 0-dimensional zero-set spaces, their realcompactifications, and their associated function spaces (see [3]).

3. Recently F. Dashiell [1] has shown that if \(X\) is an uncountable compact metric space, then for \(\alpha \neq \beta\), \(B_{\alpha}^{*}(X)\) is not linearly isometric to \(B_{\beta}^{*}(X)\), which may be thought of as strengthening the classical result that for \(\alpha < \beta\), \(B_{\alpha}^{*}(X)\) is a proper subspace of \(B_{\beta}^{*}(X)\).

A compact space is called dispersed if it contains no nonempty perfect subsets. It is known that a compact space \(X\) contains a nonempty perfect subset if and only if for each \(\alpha < \omega\), \(B_{\alpha}^{*}(X)\) is a proper subspace of \(B_{\alpha+1}^{*}(X)\), and if and only if \(B_{2}^{*}(X) \setminus B_{1}^{*}(X) \neq \emptyset\) (see [5] and [6]). Part of this also follows from the next theorem, since a nondispersed compact space admits a continuous map onto the unit interval.

**Theorem 3.** If \(X\) is a compact space, \(\alpha > 0\), \(f\) a continuous real-valued map on \(X\), and \(B \in EA_{\alpha}(f[X])\), then \(f^{-1}[B] \in EA_{\alpha}(X)\). The same holds for exactly additive and exactly multiplicative classes.

**Theorem 4.** If \(f\) is a continuous map of a compact space \(X\) onto a compact space \(Y\), then for \(\alpha = 0, 1, 2\) or \(\alpha = \omega\), \(B \in EA_{\alpha}(Y)\) implies that \(f^{-1}[B] \in EA_{\alpha}(X)\), and for \(2 < \alpha < \omega\), \(B \in A_{\alpha}(Y)\) implies that \(f^{-1}[B] \in EA_{\alpha-1}(X) \cup EA_{\alpha}(X)\). The same holds for exactly additive and exactly multiplicative classes.

**Theorem 5.** (1) Let \(X\) and \(Y\) be compact spaces and suppose that either \(X\) or \(Y\) is not dispersed. For \(0 \leq \alpha < \beta \leq \omega\), \(B_{\alpha}^{*}(X)\) is not linearly isometric to \(B_{\beta}^{*}(Y)\).

(2) If \(X\) and \(Y\) are infinite dispersed compact spaces, then \(B_{\omega}^{*}(X)\) is not linearly isometric to \(B_{2}^{*}(Y)\). (Note that \(B_{1}^{*}(X) = B_{2}^{*}(X)\).)

**Proof Outline.** (1) Suppose \(\alpha < \beta\), that \(X\) is not dispersed, and that \(B_{\alpha}^{*}(X)\) is linearly isometric to \(B_{\beta}^{*}(Y)\). Then by Theorem 2 there is a Baire isomorphism \(\phi\) of \(Y\) onto \(X\) of class \((\beta, \alpha; \beta, \alpha)\). Thus there is a ring isomorphism \(\Phi\) of \(B_{\omega}(X)\) onto \(B_{\omega}(Y)\) such that \(\Phi[B_{\alpha}(X)] = B_{\beta}(Y)\) defined by \(\Phi(h)(y) = h(\phi(y))\) for all \(h \in B_{\alpha}(X)\) and \(y \in Y\). Let \(f: X \to [0, 1]\) be a continuous map onto the unit interval. Let \(\{g_{n}: n = 1, 2, \ldots\} \subseteq C(Y)\) be such that \(\Phi(f)\) is contained in the smallest class of functions
containing \( \{g_n: n = 1, 2, \ldots \} \) and closed under pointwise sequential limits.

Consider the map \( \Psi: Y \to \mathbb{R}^N \) defined by \( \Psi(y) = (g_1(y), g_2(y), \ldots) \).

Then the Baire isomorphism \( \phi \) of \( Y \) onto \( X \) induces a Baire measurable map \( \tilde{\phi} \) of \( \Psi(Y) \) onto \([0, 1]\). There is a Cantor set \( C \subseteq \Psi(Y) \) such that \( \tilde{\phi} \) restricted to \( C \) is a homeomorphism [7, p. 444], since \( \tilde{\phi} \) is continuous apart from a set of first category [7, p. 400]. Thus there is a set \( B \in EA_\beta(\tilde{\phi}[C]) \subseteq EA_\beta([0, 1]) \). By Theorem 3, \( \Psi^{-1}[\tilde{\phi}^{-1}[B]] \in EA_\beta(Y) \) and \( f^{-1}[B] \in EA_\beta(X) \).

This implies that the characteristic function

\[
\chi_{\Psi^{-1}[\tilde{\phi}^{-1}[B]]} \in B_\beta^*(Y) \cup \bigcup_{\xi < \beta} B_\xi^*(Y).
\]

but since \( \Phi^{-1}(\chi_{\Psi^{-1}[\tilde{\phi}^{-1}[B]]) = \chi_{f^{-1}[B]} \), it follows that \( \chi_{f^{-1}[B]} \in B_\alpha^*(X) \).

This contradiction completes the proof.

(2) This follows from the fact that \( X_1 \) and \( Y_1 \) contain nonempty compact perfect subsets and \( X \) and \( Y \) do not.

4. A topological space \( X \) is called an F-space if for each disjoint pair \( C_1, C_2 \in CZ_0(X) \) there is a disjoint pair \( Z_1, Z_2 \in Z_0(X) \) such that \( C_1 \subseteq Z_1 \) and \( C_2 \subseteq Z_2 \).

**Lemma.** Let \( X \) be any topological space and \( \alpha \geq 1 \). If \( Z_1, Z_2 \in Z_\alpha(X) \) are disjoint, then there is a set \( A \in A_\alpha(X) \) such that \( Z_1 \subseteq A \subseteq X \setminus Z_2 \).

**Theorem 6.** If \( X \) is a nondispersed compact space and \( 1 < \alpha < \Omega \), then there exist disjoint sets \( C_1, C_2 \in CZ_\alpha(X) \) such that there does not exist a set \( A \in A_\alpha(X) \) with \( C_1 \subseteq A \subseteq X \setminus C_2 \). Consequently, \( X_\alpha \), the Stone space of \( A_\alpha(X) \), is not an F-space.

Remarks. (1) For any space \( X \) the Stone space of \( A_\Omega(X) = (Z_\Omega(X)) \) is an F-space.

(2) For uncountable compact metric spaces a stronger result than Theorem 5 is obtained in [1].

**References**


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