We consider the linearized Boltzmann equation

\[ \frac{\partial p}{\partial t} + \xi \cdot \text{grad} \, p = Qp/\epsilon, \]

whose solution \( p = p_\epsilon(t, x, \xi), \quad t > 0, \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad \epsilon > 0. \) \( Q \) is the linearized collision operator corresponding to a spherically symmetric hard potential, and \( \epsilon \) is a parameter which represents the mean free path.

In a series of basic papers, Grad [6], [7], [8] studied the existence and asymptotic behavior of the solution of the initial value problem for (1), where the initial data \( p_\epsilon(0^+, x, \xi) = f(x, \xi) \) satisfies mild growth and smoothness conditions. Grad's method begins with the decomposition

\[ Q = -\nu + K, \]

where \( \nu \) is the operator of multiplication by the collision frequency \( \nu(\xi), \) a strictly positive function of \( |\xi|, \) and \( K \) is a compact operator on the Hilbert space \( H_0 \) of functions \( f(\xi) \) which satisfy

\[ \langle f, f \rangle \equiv \left( \frac{1}{\sqrt{2\pi}} \right)^3 \int |f(\xi)|^2 \exp\left(-|\xi|^2/2\right) d\xi < \infty. \]

Using (2), Grad wrote (1) as an integral equation and then derived \textit{a priori} estimates for the solution in the Hilbert space

\[ H \equiv L^2(R^6, (1/\sqrt{2\pi})^3 \exp(-|\xi|^2/2) \, dx \, d\xi). \]

Grad also related the asymptotic behavior of \( p_\epsilon \) to the solutions of the linear Euler and Navier-Stokes equations. Given \( f \in H, \) define

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\[ f_0(x) = \langle f(x, \cdot), 1 \rangle; \]
\[ f_i(x) = \langle f(x, \cdot), \xi_i \rangle, \quad i = 1, 2, 3; \]
\[ f_4(x) = \langle f(x, \cdot), (|\xi|^2 - 3)/\sqrt{6} \rangle, \]
where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( H_0 \). The Navier-Stokes equations are written

\[
\frac{\partial n_0}{\partial t} + \text{div} \ n = 0, \\
\frac{\partial n}{\partial t} + \text{grad} \ n_0 + \sqrt{2/3} \ \text{grad} \ n_4 = \varepsilon \eta [\Delta n + (1/3) \ \text{grad} \ \text{div} \ n], \\
\frac{\partial n_4}{\partial t} + \sqrt{2/3} \ \text{div} \ n = \varepsilon \lambda \Delta n_4, \\
n_i(0^+, \cdot) = f_i.
\]

In (3), \( \varepsilon > 0, n_i = n_i^\varepsilon(t, x) \ (i = 0, \cdots, 4), n = (n_1, n_2, n_3), \) and \( \eta > 0 \) and \( \lambda > 0 \) are physical constants. The Euler equations are obtained from (3) by putting \( \varepsilon = 0 \). Setting

\[
p_e = T_e(t)f, \\
N_e(t)f = n_0^\varepsilon + \sum_{i=1}^{3} n_i^\varepsilon \xi_i + n_4^\varepsilon \frac{|\xi|^2 - 3}{\sqrt{6}}, \\
E(t)f = N_0(t)f,
\]
Grad proved the following asymptotic results:

\[
T_e(t)f = E(t)f + O(\varepsilon), \quad (\varepsilon \downarrow 0) \\
T_e(t/\varepsilon)f = N_e(t/\varepsilon)f + O(\varepsilon).
\]

In physical terms, (4) describes the nonviscous fluid approximation at a fixed time \( t > 0; \) (5) describes the viscous effects when \( t \to \infty \). Our aim is to show that (5) is only one of a large variety of possible refinements of (4). This is accomplished by the following two results.

**BOLTZMANN LIMIT THEOREM.** Let \( f(x, \xi) \) be sufficiently regular. Then

\[
E(-t/\varepsilon)T_e(t/\varepsilon)f = \bar{N}(t)f + O(\varepsilon) \quad (\varepsilon \downarrow 0),
\]
where \( \bar{N}(t) \) is a contraction semigroup on \( H \) whose generator is given by the differential equations
\[
\frac{\partial \eta_0}{\partial t} = \left( \frac{9}{25} \lambda + \frac{2}{5} \eta \right) \Delta \eta_0 + \sqrt{\frac{2}{3}} \left( -\frac{6}{25} \lambda + \frac{2}{5} \eta \right) \Delta \eta_4,
\]
\[
\frac{\partial \eta}{\partial t} = \eta \Delta \eta + \left( \frac{\lambda}{5} - \frac{\eta}{3} \right) \text{grad div} \, \eta,
\]
\[
\frac{\partial \eta_4}{\partial t} = \sqrt{\frac{2}{3}} \left( -\frac{6}{25} \lambda + \frac{2}{5} \eta \right) \Delta \eta_0 + \left( \frac{11}{25} \lambda + \frac{4}{15} \eta \right) \Delta \eta_4,
\]
\[
n(t)(0^+, x) = f_t(x);
\]
derived text

i.e.,
\[
\overline{N}(t)f = \eta_0 + \sum_1^3 n_i \xi_i + \eta_4 \frac{|\xi|^2 - 3}{\sqrt{6}}.
\]

The semigroup \( \{\overline{N}(t), t \geq 0\} \) commutes with the Euler semigroup \( \{E(t), t \geq 0\} \).

In order to make connection with (5) we also need the following.

**Navier-Stokes Limit Theorem.** Let \( f(x, \xi) \) be sufficiently regular.

Then
\[
E(-\frac{t}{\epsilon}) N_\epsilon(t/\epsilon)f = \overline{N}(t)f + O(\epsilon) \quad (\epsilon \downarrow 0).
\]

The proof of (8) proceeds by means of Fourier transformation from the following purely algebraic result, of independent interest.

**Matrix Limit Theorem.** Let \( A, B \) be real, symmetric \( m \times m \) matrices and assume that \( B \) is negative semidefinite. Then
\[
\exp(-itA/\epsilon) \exp(t(iA + \epsilon B)/\epsilon) = \exp(t\pi_A B) + O(\epsilon) \quad (\epsilon \downarrow 0),
\]
where \( \pi_A B \) is the orthogonal projection, in the space of \( m \times m \) matrices, of \( B \) onto the linear subspace of matrices which commute with \( A \).

In particular, we show that \( \overline{N}(t) \) is obtained by a projection, in the space of operators, of \( N_\epsilon(t) \) upon the set of operators which commute with \( \{E(t), t \geq 0\} \).

Using (6), we have
\[
T_\epsilon(t/\epsilon)f = E(t/\epsilon) \overline{N}(t)f + O(\epsilon) \quad (\epsilon \downarrow 0).
\]

This is the simplest of an infinite number of alternatives to (5). Indeed, if \( \tilde{N}(t) \) is any operator whose projection is \( \overline{N}(t) \), then we may substitute \( \tilde{N}(t) \) for \( \overline{N}(t) \) in (9).

The proof of (6) depends on a careful spectral analysis of the operator \( Q - i(\gamma \cdot \xi) \), where \( \gamma \in \mathbb{R}^3 \) is a parameter. We prove the existence and
differentiability, for $|\gamma|$ sufficiently small, of the hydrodynamical eigenvalues and eigenfunctions \{\alpha^{(j)}(\gamma), e^{(j)}(\gamma); j = 1, \cdots, 5\} which satisfy $\alpha^{(j)}(0) = 0$, $e^{(j)}(0) \in \text{span} \{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}$. We then prove a contour integral representation

$$\exp \left[ r(Q - i(\gamma \cdot \xi)) \right] f = \sum_{j=1}^{5} \exp \left( r\alpha^{(j)}(\gamma) \right) f^{(j)}(\gamma) e^{(j)}(\gamma)$$

(10)

$$+ \frac{1}{2\pi i} \int_C e^{r\alpha} R(\alpha, \gamma) \frac{(Q - i(\gamma \cdot \xi))^2}{\alpha^2} f d\alpha,$$

where $C$ is a vertical contour in the half plane $\Re \alpha < 0$ and $R(\alpha, \gamma) = (Q - i(\gamma \cdot \xi) - \alpha)^{-1}$. The first term of (10) corresponds to the Hilbert solution and gives the connection with hydrodynamics. The second term is negligible in the hydrodynamic limit. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$ ($\alpha > 0$), the contour integral may be replaced by $\int_C e^{r\alpha} R(\alpha, \gamma) f d\alpha$, where the contour $C$ is such that $\Re \alpha \to -\infty$ when $\Im \alpha \to \pm \infty$. The existence of the eigenvalues $\alpha^{(j)}(\gamma)$ follows by applying the implicit function theorem to the exact hydrodynamical dispersion laws. Previously, exact dispersion laws were obtained [11] only for hard sphere potentials, i.e., $\nu(\xi) \sim |\xi|$ as $|\xi| \to \infty$. In this case, the $\alpha^{(j)}(\gamma)$ are analytic functions and can also be obtained from Rellich’s perturbation theorem [9], [10]. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$, $0 < \alpha < 1$, the $\alpha^{(j)}(\gamma)$ will not be analytic around $\gamma = 0$. Nevertheless, we obtain an asymptotic development

$$\alpha^{(j)}(\gamma) \sim \sum_{n=1}^{\infty} \alpha_{n}^{(j)} |\gamma|^n \quad (1 \leq j \leq 5),$$

where $\alpha_{1}^{(j)}$ is imaginary and $\alpha_{2}^{(j)} < 0$. These constants can be computed by formal perturbation theory. They correspond to the adiabatic sound speed and absorption coefficients for low frequency sound waves [5].

The results (6) and (8) extend known results on finite-state velocity models in one dimension [1], [2] to the full three-dimensional linearized Boltzmann equation. These theorems are valid in any number of dimensions. Their proofs and related matters will appear in full detail in [3], [4].

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