WITT CLASSES OF INTEGRAL REPRESENTATIONS
OF AN ABELIAN p-GROUP

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1. Introduction. For a Dedekind domain, \( R \), the orthogonal and symplectic representations of a finite group, \( \pi \), on finitely-generated projective inner-product modules over \( R \) admit a Witt equivalence relation, and the resulting equivalence classes form a commutative algebra, \( \mathcal{W}(R, \pi) \), over the Witt ring of \( R \). This concept has received considerable attention recently [2], [3], [4]. Our interest is motivated by the fact that \( \mathcal{W}(\mathbb{Z}, \pi) \) is so very specifically related to the bordism classification of smooth, orientation preserving actions of \( \pi \) on closed even-dimensional manifolds. We shall discuss

\[
(1.1) \text{ Theorem. If, for } p \text{ an odd prime, } \pi \text{ is an abelian } p\text{-group then } \mathcal{W}(\mathbb{Z}, \pi) \text{ contains no torsion.}
\]

A corollary of (1.1) is that for an action \((\pi, M^{2k})\) of such a group on a closed oriented manifold, the Atiyah-Singer-Segal \(G\)-signature theorem [1] determines the integral Witt class of \((\pi, H^*(M; \mathbb{Z})/\text{tor})\) uniquely. The present techniques may also be applied to determine \( \mathcal{W}(\mathbb{Z}, \pi) \) for an abelian 2-group, however torsion is present always. Thus for an orientation preserving action \((\pi, M^{2k})\) of an abelian 2-group, a torsion valued invariant, as well as the multsignature, must be computed.

By rough analogy with [5, IV, (3.3)] there is

\[
(1.2) \text{ Lemma. For any } p\text{-group } \mathcal{W}_2(Z, \pi) \cong \mathcal{W}_2(Z(1/p), \pi),
\]

and there is a split short exact sequence

\[
0 \rightarrow \mathcal{W}_0(Z, \pi) \rightarrow \mathcal{W}_0(Z(1/p), \pi) \rightarrow \mathcal{W}(\mathbb{Z}_p) \rightarrow 0.
\]
We use the subscripts 0 and 2 respectively to denote orthogonal and symplectic representations.

2. Cyclic $p$-groups. From this point we restrict our attention to odd primes. For $n \geq 0$ we denote by $Q(\lambda)$ the $p^{n+1}$-cyclotomic extension of the rationals. In the ring of algebraic integers, $Z(\lambda)$, there is the multiplicative subset, $S$, generated by the rational prime, $p$, and $S^{-1}Z(\lambda) = D \subset Q(\lambda)$ is a Dedekind domain invariant under complex conjugation. We may thus speak of the Witt ring of Hermitian inner-product modules over $D$, $H_0(D)$, and by introducing skew-Hermitian inner-products there is $H_2(D)$ and hence an algebra, $H_*(D)$.

(2.1) Lemma. For $n \geq 0$ there is an additive isomorphism

$$\hat{w}_*(Z(1/p), Z_{p^{n+1}}) \cong \hat{w}_*(Z(1/p), Z_{p^n}) \oplus H_*(D).$$

Very briefly, we consider a $(Z_{p^{n+1}}, V)$ where $V$ is an inner-product module over $Z(1/p)$ and choose a generator $T \in Z_{p^{n+1}}$. With $\tau = Tp^n$, we introduce into $V$ a selfadjoint projection operator

$$\sum v = (v + \tau(v) + \cdots + \tau^{p-1}(v))/p.$$

This yields an orthogonal decomposition $V = I \oplus I^\perp$ into the image, $I$, of $\Sigma$ and the kernel, $I^\perp$. On $I$ the subgroup generated by $\tau$ acts trivially, so we may replace $Z_{p^{n+1}}$ by the quotient group $Z_{p^n}$. Now $D$ is the quotient of the group ring $Z(1/p)(Z_{p^{n+1}})$ by the principal ideal which $1 + \tau + \cdots + \tau^{p-1}$ generates. In this fashion $I^\perp$ becomes a projective $D$-module. The (skew-)Hermitian inner-product on $I^\perp$ is $[v, w] = \Sigma_j (v, T^j(w))\lambda^j$.

At this point standard algebraic number theory intervenes in proving

(2.2) Lemma. The Hermitian Witt ring $H_0(D)$ has no torsion.

Denoting by $Q(\lambda + \lambda^{-1})$ the subfield of real elements in $Q(\lambda)$, we may paraphrase the discussion in [5, IV, §4] to show that $v \in Q(\lambda + \lambda^{-1})^*$ can, up to multiplication by a Hermitian square, be realized as the discriminant of a Hermitian inner-product module over $D$ with even rank if and only if $LL^* = vD$ for some fractional $D$-ideal $L \subset Q(\lambda)$. The key lemma then is

(2.3) Lemma. If $LL^* = vD$ then $v$ is a Hermitian square if and only if it is positive in every ordering of $Q(\lambda + \lambda^{-1})$. 


The lemma depends on the fact that only the rational prime $p$ ramifies in $Q(\lambda)$. It then proceeds from a combination of the local norm index theorem for units [6, IX, p. 187] with the reciprocity law for Hilbert symbols in the number field $Q(\lambda + \lambda^{-1})$. As a consequence of Landherr [5, p. 118, Example 4], this lemma eliminates torsion in $H_0(D)$. It is possible to determine $H_0(D)$ completely. It is necessary to produce elements $v \in Q(\lambda + \lambda^{-1})^*$ with arbitrarily prescribed signs and satisfying $LL^{-1} = vD$. This cannot be accomplished, in general, by only selecting units in $D^* \cap Q(\lambda + \lambda^{-1})^*$, and the argument involves a careful analysis of the role of the homology groups of $Z_2$ acting on the ideal class group of $D$ via conjugation of fractional $D$-ideals.

Beginning with $n = 0$, Lemmas (2.2), (2.1) and (1.2) are combined inductively to yield (1.1) for the cyclic $p$-groups. Since $D^*$ contains an imaginary unit, $H_0(D) \simeq H_2(D)$ additively so there is no special problem in handling the symplectic case.

3. The general case. We express $\pi$ as a direct sum $\pi = \pi_1 \oplus Z_{p^{n+1}}$ with $n$ as large as possible. Now proceeding as in (2.1) we split $W_*(Z(1/p), \pi)$ into a direct sum $W_*(Z(1/p), \pi_1 \oplus Z_{p^n}) \oplus H_*(D, \pi_1)$. From our choice of $n$ we may identify the character group $\pi_1^*$ with $\text{Hom}(\pi_1, D^*)$. It follows readily that $H_*(D, \pi_1) \simeq Z(\pi_1^*) \oplus H_*(D)$. Combining (2.1) and (1.2) with this observation, (1.1) is established.

For a specific example if $p$ is odd, the multisignature is an isomorphism of $W_*(Z, Z_p)$ onto the subring of the group ring $Z(Z_p)$ consisting of those elements which can be expressed in the form

$$m_0 \cdot e + \sum_{i=1}^{k} m_i (\tau^i + \tau^{-i}) + \sum_{i=1}^{k} n_i (\tau^i - \tau^{-i}),$$

where $k = (p - 1)/2$ and the integral coefficients satisfy $m_1 = m_2 = \cdots = m_k \pmod{2}$ and $n_1 = n_2 = \cdots = n_k \pmod{2}$.

ADDED IN PROOF. Theorem (1.1) can be proven for general $p$-groups $\pi$, for example by the induction techniques of Dress [2].

REFERENCES


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