THE HOPF RING FOR COMPLEX COBDISM

BY DOUGLAS C. RAVENEL AND W. STEPHEN WILSON

Communicated by Edgar Brown, Jr., May 13, 1974

It is our purpose here to announce the results of our study of the homology of the spaces in the \(\Omega\)-spectrum for complex cobordism and Brown-Peterson cohomology. Let \(MU(n)\) be the standard Thom complex. \(MU_k = \lim_{n \to \infty} \Omega^{2(n-k)}MU(n)\) is the \(2k\) space in the \(\Omega\)-spectrum for complex cobordism. We will consider the space \(MU = \lim_{n \to \infty} \Pi_{j \geq n} MU_j\). We find this product easier to study than the separate factors, as will become apparent below.

For a space \(X\) we have \([X, MU] \cong U^2*(X)\), the even degree part of the complex cobordism of \(X\). Because \(MU\) is a multiplicative theory, \(U^2*(X)\) is a ring and \(MU\) is a commutative ring with identity in the homotopy category. Thus we have that for any field \(k\), \(H_*(MU; k)\) is a commutative ring with identity in the category of \(k\)-coalgebras, i.e., it is a "Hopf ring".

In more common language, the homology has two products and a coproduct. \(\circ\) will denote the multiplicative product which comes from the ring structure on the spectrum, while \(\ast\) will denote the additive product coming from the loop structure \((\Omega^2MU \simeq MU)\). They obey the following distributive law: if \(\psi(z) = \Sigma z' \otimes z''\) is the coproduct, then \(z \circ (x \ast y) = \Sigma (z' \circ x) \ast (z'' \circ y)\).

We now describe the structure of \(H_*(MU; R)\) where \(R\) is an algebra over a field \(k\). Let

\[
C_R(X) = \left\{ x \in \prod \limits_{i \geq 0} H_i(X; R) : \psi(x) = x \hat{\otimes} x, x \neq 0 \right\}
\]

\(C_R(MU)\) is a ring, and for each \(x \in C_R(X)\) we have a ring homomorphism \(\lambda_x: U^2*(X) \to C_R(MU)\) defined by \(\lambda_x(u) = u_*(x)\) for \(u \in U^2*(X)\). Let

---

AMS (MOS) subject classifications (1970). Primary 55F40, 57D90, 57F25, 57F35; Secondary 57F05, 55F20, 16A24, 14L05, 18D35.

Key word and phrases. Complex cobordism, homology, Hopf ring, formal group law, \(\Omega\)-spectrum, Eilenberg-Moore spectral sequence, classifying space.

Both authors partially supported by NSF.
$\beta_n \in H_{2n}(\mathbb{CP}^\infty; R)$ be the standard generator. $\beta(r) = \Sigma \beta_i r^i \in C_R(\mathbb{CP}^\infty)$ for $r \in R$.

$U^2*\mathbb{CP}^\infty = U^*[\langle T \rangle]$, the power series ring on the canonical generator $T \in U^2\mathbb{CP}^\infty$ over the coefficient ring $U^* = Z[x_2, x_4, \ldots]$, a polynomial algebra on negative even-dimensional generators. We now have

$$b(r) = \sum b_i r^i = \lambda_{\beta(r)}(T) \in C_R(MU).$$

(In other words, if we represent $T$ by a map $f: \mathbb{CP}^\infty \to MU, f_\ast(\beta_n) = b_n$.)

Note that $\pi_0 MU \simeq \pi_* MU \simeq U_* \simeq U^{-*}$, and any element $a \in U^*$ or $U_*$ gives rise to an element $[a] \in H_0 MU$. The $[x_{2i}]$ generate the Hopf ring $H_0 MU$. Under the standard multiplication $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, $T$ pulls back to $\Sigma a_{ij} T_1^i \otimes T_2^j$, where $a_{ij} \in U^{2(1-i-j)}$. The $a_{ij}$ are the coefficients of the formal group associated with complex cobordism (see [1]).

We use the above multiplication to get our first theorem.

**Theorem 1.** In $C_R(MU)$,

$$b(r_1 + r_2) = \sum_{i,j>0} [a_{ij}] b(r_1)^i b(r_2)^j.$$  

The following is just a restatement of the theorem.

**Theorem 1’.** In $H_\ast(MU; R)$,

$$b(r_1 + r_2) = \sum_{i,j>0} ([a_{ij}] \circ b(r_1)^i \circ b(r_2)^j).$$

**Corollary 2.** $\log b(r_1 + r_2) = \log b(r_1) + \log b(r_2)$ in $C_R(MU) \otimes Q$, where

$$\log X = \sum_{n>0} \frac{[\mathbb{CP}^{n-1}]}{n} X^n.$$  

If we are working over the integers we can rephrase this to:

**Corollary 2’.** $\log b(r) = b_1 r$ in $QH_\ast(MU; Q[r])$.

Let $H_R MU$ denote the Hopf ring generated by the $[x_{2i}]$ and the $b_n$ subject to the relations implied by Theorem 1.

**Theorem 3.** The map $H_R MU \to H_\ast(MU; R)$ is a Hopf ring isomorphism.

This is still true if we replace $R$ by $Z$.  


The main result of [6], where the investigation of the homology of $MU_k$ was begun, is now an immediate corollary of Theorem 3.

**Corollary 4.** $H_*(MU_k; \mathbb{Z})$ has no torsion.

**Proof.** $H_R MU$ has only even-dimensional elements.

Theorem 3 is a total information result. Not only does it give a complete description of both products and the coproduct, but, using the results of Switzer [5] on the coaction of the dual of the Steenrod algebra on $CP^\infty$, we can compute the structure of $H_*(MU; F_p)$ as a comodule over the dual to the Steenrod algebra directly from our algebraic construction $H_R MU$.

The most difficult part of the proof of Theorem 3 is showing that the map is onto. To do this, we first replace $MU$ by $BP$, the Brown-Peterson spectrum [2], [3]. We can recover information about $MU$ from $BP$ by Quillen's result that $U^*(X)_{(p)} \simeq U^*_{(p)} \otimes_{BP^*} BP^*(X)$. There are, of course, analogues of Theorems 1–4 for the analogous space $BP$. We have

$$H_*(MU; F_p) \simeq H_0(MU; F_p) \otimes_{H_0(BP; F_p)} H_*(BP; F_p)$$

and $BP_* \simeq \pi_* BP \simeq Z_{(p)}[v_1, v_2, \ldots]$, where $v_s$ is a $2(p^s - 1)$-dimensional generator. From now on, all homology groups will have coefficients in $F_p$. An immediate consequence of Theorem 1 is that all the $b_i$ can be expressed in terms of $b_{p^n}$, which we denote by $b_{(n)}$. Note that these elements generate the stable homology $H_* BP$. Define

$$v^I b^J = [v_1^{i_1} v_2^{i_2} \cdots] \circ b_{(0)}^{o j_0} \circ b_{(1)}^{o j_1} \circ \cdots,$$

where $I = (i_1, i_2, \ldots)$ and $J = (j_0, j_1, \ldots)$ are sequences of nonnegative integers, and $b_{(n)}^{o j_n}$ denotes the $j_n$th power of $b_{(n)}$ under the multiplicative or $\circ$ product.

**Definition.** $v^I b^J$ is called allowable if

$$J = p \Delta_{k_1} + p^2 \Delta_{k_2} + \cdots + p^n \Delta_{k_n} + J' \text{ (nonneg. seq.),}$$

$$k_1 \leq k_2 \leq \cdots \leq k_n$$

implies $i_n = 0$. ($\Delta_k$ is the sequence with 1 in the $k$th place and zeros elsewhere.)

Let $BP_{(0)}$ denote the zero component of $BP$. $H_* BP_{(0)}$ is a Hopf algebra under the $*$ product. Let $Q$ and $P$ denote the indecomposables and primitives respectively. We now have
THEOREM 5. (a) $H^*_{BP(0)}$ is a polynomial algebra.
(b) The allowable $v^J b^J$ ($J \neq 0$) form a basis for $QH^*_{BP(0)}$.
(c) The $v^J b^J + \Delta^0$ with $v^J b^J$ allowable ($J$ possibly zero) form a basis for $PH^*_{BP(0)}$.

The proof of Theorem 5 is by induction on dimension, using Eilenberg-Moore spectral sequences which go from $H^*_{BP(0)}$ to $H^*_{\Omega BP(0)}$ and back to $H^*_{BP(0)}$ using the periodicity $\Omega^2 BP(0) \cong BP$ and Theorem 6.

$H^*_{BP}$ is a $BP^*$ module under the $\circ$ product as $BP^*CH_0$. We have the ideal $(u_1, u_2, \ldots) = I \subset BP_*$. The $[p]$-sequence $[p](X)$ can be defined by $\log_{BP}[p](X) = p \log_{BP}(X)$. Also, $[p](T)$ is the image of $T$ when pulled back by the $p$th power map $CP^\infty \rightarrow (CP^\infty)^p \rightarrow CP^\infty$ in $BP^*CP^\infty \cong BP^*[[T]]$. (Note. $b = b(1)$.)

THEOREM 6. (a) $[p](b) = 0$ in $C_{Fp}(BP)$.
(b) $\sum_{i=1}^n [v_i] \circ b_{(n-i)}^p = 0$ in $QH^*_{BP}/I^2 QH^*_{BP}$.

The first statement follows from the fact that the $p$th power map is trivial in $H^*_{CP^\infty}$. (Recall that our coefficients are all $F_p$.) The second statement follows from the fact that the coefficient of $X^{p^n}$ in the $[p]$ sequence is a $2(p^n - 1)$-dimensional generator in $BP_*$.

We now state some of the geometric corollaries which follow from our work.

$U_*MU$ can be identified with the cobordism group of maps (with even codimension) of compact stably almost complex manifolds (see Stong [4] for the analogous statement in the unoriented case). From this point of view our main result is

THEOREM 7. $U_*MU$ is a Hopf ring generated by maps to a point, identity maps, and linear embeddings $b_n: CP^{n-1} \hookrightarrow CP^n$.

COROLLARY 8. Any map of compact stably almost complex manifolds is cobordant to one of the form $f: \coprod F_i \times V_i \rightarrow M$, where $f|F_i \times V_i$ is the composition of the projection $F_i \times V_i \rightarrow V_i$ and an embedding $V_i \hookrightarrow M$.

REFERENCES

1. J. F. Adams, Quillen's work on formal groups and complex cobordism, Lecture Notes, University of Chicago, 1970.


DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

Current address (both authors): School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540