A REMARK ON SIMPLY-CONNECTED 3-MANIFOLDS

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In this note we will describe an "infinite process", related to "wild topology", with applications to closed, smooth manifolds.

DEFINITION. Consider an increasing sequence of solid tori:

\[ T_1 \subset T_2 \subset \cdots \subset T_n \subset T_{n+1} \subset \cdots \]

\((T_i = k(i) \# (S^1 \times D^2))\) such that:

(a) \(T_k \subset \text{int} T_{k+1}\), and \(T_k\) is a smooth submanifold of \(T_{k+1}\).

(b) The natural inclusion \(T_k \rightarrow T_{k+1}\) is null-homotopic.

The open 3-manifold \(W = \lim T_i\) will be called a Whitehead manifold.

It is an easy (and well-known) exercise to show that for any Whitehead manifold \(W\), one has: \(W \times R = R^3\).

THEOREM 1. Let \(X\) be a smooth 3-manifold with \(\pi_1X = 0\), \(T\) a solid torus, and \(T \rightarrow X\) a smooth embedding. There exists a Whitehead manifold \(W\) defined by a sequence of nested tori:

\[ T = T_1 \subset T_2 \subset \cdots \]

and a smooth embedding \(W \rightarrow X\) such that the following diagram is commutative:

\[ T \xrightarrow{j_1} T_2 \xrightarrow{j_2} \cdots \]

\[ T \xrightarrow{j} X \]

\[ \cdots \xrightarrow{j_2} \xrightarrow{j_1} W \]

PROOF. It suffices to show that there exists a solid torus \(T_2\) and a commutative diagram of smooth embeddings:
such that \( j_1 \) is null-homotopic. (Afterwards we can continue this process indefinitely.)

Consider a wedge (bouquet) of circles \( K = \bigvee_{i=1}^{p} S^1_i \) which is a spine of \( T \). There exists a commutative diagram:

\[
\begin{array}{ccc}
T & \xrightarrow{j} & X \\
j_1 & \searrow & \\
& T_2 & \\
\end{array}
\]

where \( \psi \) is a generic immersion, without triple points, such that the set of double points is a union of (disjoined) arcs.

Take \( T_2 = \) a regular neighborhood of \( \psi(\bigvee_{i=1}^{p} D^2_i) \) in \( X \) (containing \( T \) in its interior) a.s.o.

**Corollary 2.** Let \( \Sigma^3 \) be a smooth homotopy 3-sphere. There exist two open subsets \( U_1, U_2 \subset \Sigma^3 \times \mathbb{R} \) such that:

(a) \( \Sigma^3 \times R = U_1 \cup U_2 \),

(b) \( U_i \) is diffeomorphic to \( \mathbb{R}^4 \).

**Proof.** Let \( \Sigma^3 = T' \cup T'' \) be a Heegaard decomposition, and consider the Whitehead manifolds \( W', W'' \) (containing \( T', T'' \) and contained in \( \Sigma^3 \)) provided by Theorem 1.

Since \( W \times R = \mathbb{R}^4 \) for any Whitehead manifold, we can take \( U_1 = W' \times R \subset \Sigma^3 \times \mathbb{R}, \ U_2 = W'' \times R \subset \Sigma^3 \times \mathbb{R} \). Q.E.D.

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