ISOMETRIC MINIMAL IMMERSIONS OF $S^3(a)$ IN $S^N(1)$

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Introduction. We denote by $S^p(a)$ the sphere of radius $a$ in the euclidean $(p + 1)$-space $E^{p+1}$, with the induced metric. In [1], S. S. Chern asks the following question: "Let $S^3(a) \to S^7(1)$ be an isometric minimal immersion. Is it totally geodesic?". In this note we announce the following result.

THEOREM. Let $S^3(a) \subset E^4 \to S^N(1) \subset E^{N+1}$ be an isometric minimal immersion which is not totally geodesic. Then $N \geq 8$.

The class of isometric minimal immersions of $S^p(a) \to S^N(1)$ was qualitatively described by M. do Carmo and N. R. Wallach in [3]. For $p = 2$, each admissible $a$ determines a unique element of such a class. The main result of [3] shows that for each $p \geq 3$ and each admissible $a \geq \sqrt{8}$, there exists a continuum of distinct such immersions. Our Theorem is an answer to a question of quantitative character. This constitutes part of our doctoral dissertation at IMPA. I want to thank my adviser Professor M. do Carmo for suggesting this problem and for helpful conversations.

Definitions and lemmas. Let $H = (\varphi_0, \ldots, \varphi_N): S^3(a) \subset E^4 \to S^N(1) \subset E^{N+1}$ be an isometric minimal immersion. Then [1] the coordinate functions are spherical harmonics on $S^3(a)$, i.e., each $\varphi_i$ $(0 \leq i \leq N)$ is the restriction to $S^3(a)$ of a homogeneous polynomial of degree $s$, with four indeterminates satisfying the condition

\[
\sum_{k=1}^{4} \frac{\partial^2 \varphi_i}{\partial x_k^2} = 0.
\]

Initially we set

\[
\varphi_i = \sum_{\sum \alpha_i = s} a_{\alpha_1} \cdots a_{\alpha_4} x_1^{\alpha_1} \cdots x_4^{\alpha_4},
\]

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and write

$$H = \sum_{\alpha_1=1}^{s} A_{\alpha_1} \cdots A_{\alpha_4} x_1^{\alpha_1} \cdots x_4^{\alpha_4},$$

where the vectors $A_{\alpha_1} \cdots A_{\alpha_4} = (a_{\alpha_1}^0 \cdots a_{\alpha_4}^N, \cdots, a_{\alpha_1}^N \cdots a_{\alpha_4}^N) \in E^{N+1}$ are the column-vectors of the matrix in which the $i$th row is made up of the coefficients of $\varphi_i (0 \leq i \leq N)$. We denote by $V(H)$ the subspace of $E^{N+1}$ generated by the vectors $A_{\alpha_1} \cdots A_{\alpha_4}$.

Identify the set

$$X^s = \left\{ x_1^{\alpha_1} \cdots x_4^{\alpha_4}; \sum \alpha_i = s, \alpha_i \geq 0, \text{ integer} \right\}$$

with the tetrahedron

$$T^s = \left\{ (\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) \in E^{N+1}; \sum \alpha_i = s, \alpha_i \geq 0, \text{ integer} \right\}$$

by means of the correspondence

$$x_1^{\alpha_1} \cdots x_4^{\alpha_4} \leftrightarrow (\alpha_1, \cdots, \alpha_4, 0, \cdots, 0).$$

It can be shown that the equivalence relation defined on $T^s$: $(\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) \in T^s$ and $(\alpha'_1, \cdots, \alpha'_4, 0, \cdots, 0) \in T^s$ are equivalent iff for each $i = 1, \cdots, 4$, $\alpha_i - \alpha'_i \equiv 0 \pmod{2}$ decomposes $T^s$ in 8 equivalence classes $C_i, i = 1, \cdots, 8$, if $s > 2$. Let $\pi: T^s \rightarrow E^{N+1}$ be the map

$$\pi(\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) = A_{\alpha_1} \cdots A_{\alpha_4},$$

and denote by $V_i(H)$ the subspace of $E^{N+1}$ generated by $\pi(C_i), i = 1, \cdots, 8$.

With the above notation we have

**Lemma A.** Let $H = \Sigma A_{\alpha_1} \cdots A_{\alpha_4} x_1^{\alpha_1} \cdots x_4^{\alpha_4}$ be an isometry of $S^3(a)$ in $S^N(1)$ where $H$ is given by (3). Then $V(H) = V_1(H) \oplus \cdots \oplus V_8(H)$, if $\dim V_i(H) = 1, i = 1, \cdots, 8$.

**Lemma B.** Let $H = (\varphi_0, \cdots, \varphi_N): S^3(a) \subset E^4 \rightarrow S^N(1) \subset E^{N+1}$ be an isometric minimal immersion which is not totally geodesic. Then the dimension of $V(H)$ is $> 8$.

The Theorem follows immediately from Lemmas A and B.

**Sketches of proofs.** First we establish some general facts. We indicate the inner product of $A, B \in E^{N+1}$ by $AB$ (or $A^2$ if $A = B$).
Under the conditions of Lemma A, we have

\[ H^2 = \sum_{\alpha_i = s; \beta_i = s} A_{\alpha_1 \cdots \alpha_4} A_{\beta_1 \cdots \beta_4} x_1^{\alpha_1 + \beta_1} \cdots x_4^{\alpha_4 + \beta_4} , \]

and

\[ 1 = a^{-2s} \sum_{\alpha_i = s} \frac{s!}{\alpha_1! \cdots \alpha_4!} x_1^{2\alpha_1} \cdots x_4^{2\alpha_4} . \]

Under the conditions of Lemma B, we have that the coordinate functions \( \varphi_i, i = 0, \ldots, N, \) are spherical harmonics. This implies for each element in \( T^{g-2} \) a linear relation of the type

\[ \alpha_1 (\alpha_1 - 1) A_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} + (\alpha_2 + 2)(\alpha_2 + 1) A_{(\alpha_1 - 2)(\alpha_2 + 2)\alpha_3 \alpha_4} \]

\[ + (\alpha_3 + 2)(\alpha_3 + 1) A_{(\alpha_1 - 2)\alpha_2 (\alpha_3 + 2)\alpha_4} \]

\[ + (\alpha_4 + 2)(\alpha_4 + 1) A_{(\alpha_1 - 2)\alpha_2 \alpha_3 (\alpha_4 + 2)} = 0. \]

Then using the fact that \( H \) preserves inner products, we compare the products of tangent vectors at convenient paths on \( S^3(a) \) with those at the transformed paths on \( H(S^3(a)) \). This yields relations between the vectors \( A_{\alpha_1 \cdots \alpha_4} \). We exhibit some typical relations.

\[ A_{s000}^2 = a^{-2s}; \quad A_{(s-1)100}^2 = a^{2(1-s)}; \quad A_{s000} A_{(s-2)200} < 0; \]

\[ A_{(s-2)110}^2 + 2A_{(s-1)100} A_{(s-3)120} = (s-1)a^{2(1-s)}; \]

\[ 2A_{(s-2)200}^2 + 3A_{(s-1)100} A_{(s-3)300} = 6^{-1}s(s-1)(s+4)a^{-2s}; \]

\[ 2A_{(s-2)200} A_{(s-3)200} + A_{(s-1)100} A_{(s-3)120} = -6^{-1}s^2(s-1)a^{-2s}. \]

Similar relations are obtained for each permutation of 2 indices in \( A_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \).

Lemma A. From the definitions of \( V(H) \) and \( V_j(H), i = 1, \ldots, 8, \) we have

\[ V(H) = V_1(H) + \cdots + V_8(H). \]

The lemma follows immediately from the fact that if \( A_{\alpha_1 \cdots \alpha_4} \in V_i(H) \) and \( A_{\beta_1 \cdots \beta_4} \in V_j(H), \) with \( i \neq j, \) then

\[ A_{\alpha_1 \cdots \alpha_4} A_{\beta_1 \cdots \beta_4} = 0. \]

In order to show this, we first observe that \( \alpha_k + \beta_k \) is odd for some \( k = 1, \ldots, 4 \) (this follows from the definition of \( V_i(H) \)).

Next we use (4), (5) and \( H^2 = 1, \) and observe that the terms in (5) with odd exponents are zero. This proves our claim.
Lemma B. This is the crucial point of the proof. The proof of Lemma B is reduced to obtaining estimates for the dimensions of $V_i(H)$, $i = 1, \ldots, 8$. Such estimates follow from the study of the relations between the vectors $A_{\alpha_1 \ldots \alpha_4}$, obtained from (6) and (7). We first show that if $s$ is odd (even), then four (seven) of the subspaces $V_i(H)$ necessarily have dimension $\geq 1$. Next we examine the hypothesis of nullity of some $V_i(H)$ and conclude that in all cases the sum of the dimensions of the $V_i(H)$, $i = 1, \ldots, 8$, is greater than 8. The final and more delicate case occurs when we assume that the dimension of each $V_i(H)$, $i = 1, \ldots, 8$, is equal to 1. This assumption forces $H$ to be totally geodesic, hence, a contradiction.

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